Lecture 1: Simple Continued Fractions

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1. Introduction

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Now note that it is "self-replicating": We have

$$x = 1 + \frac{1}{x}.$$

Multiplying both sides by x, we obtain

$$x^2 - x - 1 = 0,$$

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This is the *golden mean* (or *golden ratio*), which is closely related to the *Fibonacci numbers*.

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For representing (i.e., just “writing down") arbitrary real numbers, continued fractions present an alternative to the decimal representation.

2. Finite continued fractions

Given the fraction $\frac{u_0}{u_1}$ with $(u_0, u_1) = 1$ and $u_1 > 0$, we use the Euclidean algorithm

$$\begin{aligned}
u_0 &= u_1 a_0 + u_2, && 0 < u_2 < u_1 \\
u_1 &= u_2 a_1 + u_3, && 0 < u_3 < u_2 \\
&\vdots \\
u_{j-1} &= u_j a_{j-1} + u_{j+1}, && 0 < u_{j+1} < u_j \\
u_j &= u_{j+1} a_j.
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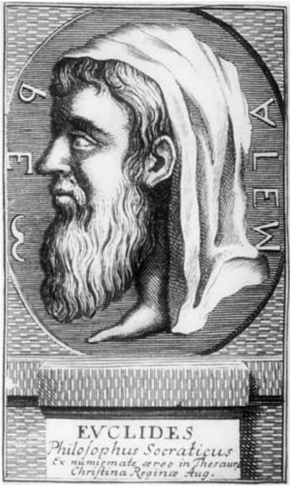
If we write $q_i := \frac{u_i}{u_{i+1}}$, $0 \leq i \leq j$, then

$$\begin{aligned} q_i &= a_i + \frac{1}{q_{i+1}}, \quad 0 \leq i \leq j-1, \\ q_j &= a_j. \end{aligned}$$



Euclid of Alexandria
($\sim$325–265 BCE)

Example: Start with $67/24$. Then

$$\begin{aligned} 67 &= 2 \cdot 24 + 19, & \tfrac{67}{24} &= 2 + \tfrac{19}{24}; \\ 24 &= 1 \cdot 19 + 5, & \tfrac{24}{19} &= 1 + \tfrac{5}{19}; \\ 19 &= 3 \cdot 5 + 4, & \tfrac{19}{5} &= 3 + \tfrac{4}{5}; \\ 5 &= 1 \cdot 4 + 1, & \tfrac{5}{4} &= 1 + \tfrac{1}{4}. \end{aligned}$$

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Substituting each equation in the second column into the one above, we obtain the expression

$$\frac{67}{24} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$

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$$\frac{67}{24} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$

Such an expression is called a *finite continued fraction* (C.F.).

In general,

$$\frac{u_0}{u_1} = q_0 = a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_{j-1} + \cfrac{1}{a_j}}} \tag{2}$$

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Notation: We denote the C.F. (2) by

$$\langle a_0, a_1, \dots, a_j \rangle.$$

(Different notations can be found in the literature).

Remarks. (a) We have

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(b) Generalizing the C.F. (2):
Let $x_0, x_1, \ldots, x_j$ be any *real* numbers, $x_1, \ldots, x_j > 0$.

Then we can define

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If all the x_i are positive integers (with x_0 allowed to be negative or zero), the C.F. is said to be **simple**.

In these lectures: **Only simple** continued fractions.

3. Uniqueness of finite continued fractions

Example:

$$\frac{51}{22} = 2 + \cfrac{1}{3 + \cfrac{1}{7}} = 2 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{1}}}$$

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Theorem 1

If

$$\langle a_0, a_1, \ldots, a_j \rangle = \langle b_0, b_1, \ldots, b_n \rangle$$

are simple C.F.s and if $a_j > 1$ *and* $b_n > 1$
then $j = n$ *and* $a_i = b_i$ *for* $i = 0, 1, \ldots, n$.

Proof. Set $y_i := \langle b_i, b_{i+1}, \dots, b_n\rangle$ and observe that

$$y_i = b_i + \frac{1}{\langle b_{i+1}, b_{i+2}, \dots, b_n\rangle} = b_i + \frac{1}{y_{i+1}}.$$

Then proceed by induction (Exercise). ■

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Consequence:

Theorem 2

Any finite C.F. represents a rational number.

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Proof. (a) Use the formula (3) repeatedly; it terminates after a finite number of steps.

(b) Second part follows from Euclid's algorithm.
Uniqueness follows from Theorem 1. ■

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Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of integers, all positive, with the possible exception of a_0.

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Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of integers, all positive, with the possible exception of a_0.

We define the two sequences $\{h_n\}$, $\{k_n\}$ by

$$h_{-2} = 0, \quad h_{-1} = 1, \quad h_i = a_i h_{i-1} + h_{i-2} \quad (i \geq 0); \tag{4}$$

$$k_{-2} = 1, \quad k_{-1} = 0, \quad k_i = a_i k_{i-1} + k_{i-2} \quad (i \geq 0). \tag{5}$$

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Note: $k_0 = 1$, $k_1 = a_1 k_0 \geq k_0$, $k_2 > k_1$, $k_3 > k_2, \ldots$.
In other words,

$$1 \leq k_1 < k_2 < k_3 < \ldots < k_n < \ldots \tag{6}$$

With the help of these two recurrence sequences we can write a finite C.F. as a plain fraction:

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Theorem 3

For any $x \in \mathbb{R}$, $x > 0$, we have

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Proof by induction. For $n = 0$, statement reduces to

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Proof by induction. For $n = 0$, statement reduces to

$$x = \frac{xh_{-1} + h_{-2}}{xk_{-1} + k_{-2}},$$

which follows immediately from (4) and (5). For $n = 1$, result is

$$\langle a_0, x \rangle = \frac{xh_0 + h_{-1}}{xk_0 + k_{-1}};$$

again easy to verify, using $\langle a_0, x \rangle = a_0 + \frac{1}{x}$.

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$$\begin{aligned}\langle a_0, a_1, \ldots, a_{n-1}, a_n, x\rangle &= \langle a_0, a_1, \ldots, a_{n-1}, a_n + \tfrac{1}{x}\rangle \\ &= \frac{\left(a_n + \frac{1}{x}\right) h_{n-1} + h_{n-2}}{\left(a_n + \frac{1}{x}\right) k_{n-1} + k_{n-2}} \\ &= \frac{x(a_n h_{n-1} + h_{n-2}) + h_{n-1}}{x(a_n k_{n-1} + k_{n-2}) + k_{n-1}} \\ &= \frac{x h_n + h_{n-1}}{x k_n + k_{n-1}},\end{aligned}$$

where we have used the recurrence relations (4) and (5). This completes the proof by induction. ■

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&= \frac{x h_n + h_{n-1}}{x k_n + k_{n-1}},
\end{aligned}$$

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As a special case we obtain:

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Proof: Use Theorem 3, and replace x by a_n:

$$r_n := \langle a_0, a_1, \ldots, a_n \rangle = \frac{a_n h_{n-1} + h_{n-2}}{a_n k_{n-1} + k_{n-2}} = \frac{h_n}{k_n},$$

where the last equation follows again by (4) and (5). ■

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(i) For $i \geq 1$,

$$h_i k_{i-1} - h_{i-1} k_i = (-1)^{i-1}, \qquad (7)$$

$$r_i - r_{i-1} = \frac{(-1)^{i-1}}{k_i k_{i-1}}. \qquad (8)$$

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Proof: By induction (Exercise).

These identities are generalizations of identities for Fibonacci numbers:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

Leonardo of Pisa (Fibonacci)
($\sim$1175–1250 BCE)

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Proof: By (8) and (10) we have

$$r_{2j} < r_{2j+2}, \quad r_{2j-1} > r_{2j+1}, \quad r_{2j} < r_{2j-1},$$

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since $k_i > 0$ for $i \geq 0$ and $a_i > 0$ for $i \geq 1$.Hence

$$r_0 < r_2 < r_4 < \ldots, \quad \text{and} \quad r_1 > r_3 > r_5 > \ldots$$

and also

$$r_{2n} < r_{2n+2j} < r_{2n+2j-1} \leq r_{2j-1}.$$

This proves part (i).

(ii) $\{r_0, r_2, \ldots\}$ is monotonically increasing and bounded above by r_1; hence it has a limit.

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Definition 7

Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of integers with $a_j > 0$ for $j \geq 1$. Then the *infinite simple* C.F. $\langle a_0, a_1, a_2, \ldots \rangle$ is defined by

$$\lim_{n\to\infty} \langle a_0, a_1, \ldots a_n \rangle.$$

Remark. The rational number

$$\langle a_0, a_1, \ldots a_n \rangle = r_n = \frac{h_n}{k_n}$$

is called the *n*th **convergent** of the infinite C.F.
(This is also defined for finite C.F.s.)

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The value of any infinite simple C.F. $\langle a_0, a_1, a_2, \ldots \rangle$ is irrational.

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Theorem 8

The value of any infinite simple C.F. $\langle a_0, a_1, a_2, \ldots \rangle$ is irrational.

Proof: Let $\theta := \langle a_0, a_1, a_2, \ldots \rangle$. Then by Theorem 6 we have

$$r_n < \theta < r_{n+1} \quad \text{or} \quad r_n > \theta > r_{n+1}.$$

Subtracting r_n, we obtain

$$0 < |\theta - r_n| < |r_{n+1} - r_n|,$$

. . . and multiplying by k_n,

$$0 < |k_n\theta - h_n| < k_n|r_{n+1} - r_n| = \frac{k_n}{k_n k_{n+1}} = \frac{1}{k_{n+1}}, \qquad (11)$$

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To obtain a contradiction, assume that $\theta \in \mathbb{Q}$ is rational, say $\theta = \frac{a}{b}$, with $a, b \in \mathbb{Z}$, $b > 0$. Now multiply (11) by b:

$$0 < |k_n a - h_n b| < \frac{b}{k_{n+1}}.$$

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$$0 < |k_na - h_nb| < \frac{b}{k_{n+1}}.$$

By (6), the integers k_n become arbitrarily large, hence $\frac{b}{k_{n+1}} < 1$ if n is sufficiently large. So finally,

$$0 < |k_na - h_nb| < 1;$$

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impossible since the term in the middle is always an integer. Our assumption was therefore false, and θ is irrational. ∎

Lemma 9

If $\theta := \langle a_0, a_1, a_2, \ldots \rangle$ *then* $a_0 = \lfloor \theta \rfloor$.
If $\theta_1 := \langle a_1, a_2, \ldots \rangle$ *then* $\theta = a_0 + \frac{1}{\theta_1}$.

Lemma 9

If $\theta := \langle a_0, a_1, a_2, \ldots \rangle$ *then* $a_0 = \lfloor \theta \rfloor$.
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Proof: By Theorem 6 we have $r_0 < \theta < r_1$, which means

$$a_0 < \theta < a_0 + \frac{1}{a_1}.$$

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For the second statement, we use an earlier remark:

$$\begin{aligned}
\theta &= \lim_{n\to\infty} \langle a_0, a_1, a_2, \ldots a_n \rangle \\
&= \lim_{n\to\infty} \left(a_0 + \frac{1}{\langle a_1, a_2, \ldots a_n \rangle} \right) \\
&= a_0 + \frac{1}{\lim_{n\to\infty} \langle a_1, a_2, \ldots a_n \rangle} = a_0 + \frac{1}{\theta_1}.
\end{aligned}$$

■

The next two results deal with questions of uniqueness.

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Two distinct infinite simple C.F.s converge to different values.

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Example: $x = x_0 = \sqrt{6}$.

We give two methods, the first one more numerical. By consecutive subtracting and taking the reciprocal, we find

$$\begin{aligned}
\sqrt{6} \simeq 2.4494897 &= 2 + 0.4494897 \\
&= 2 + \frac{1}{2.2247449} \\
&= 2 + \frac{1}{2 + 0.2247449} \\
&= 2 + \cfrac{1}{2 + \cfrac{1}{4.4494897}} \\
&= 2 + \cfrac{1}{2 + \cfrac{1}{4 + 0.4494897}}
\end{aligned}$$

and now we see that we have repetition, so that

$$\sqrt{6} = \langle 2, 2, 4, 2, 4, \ldots \rangle = \langle 2, \overline{2, 4} \rangle.$$

However, it should be noted that rounding errors will be a serious problem after only a few steps, so that the result obtained above has to be verified with the method used earlier.

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This problem is avoided if we use a different method:

$$\begin{aligned}
\sqrt{6} &= 2 + (\sqrt{6} - 2); \\
\frac{1}{\sqrt{6} - 2} &= \frac{\sqrt{6} + 2}{6 - 4} = 2 + \left(\frac{1}{2}\sqrt{6} - 1\right); \\
\frac{1}{\frac{1}{2}\sqrt{6} - 1} &= \frac{\frac{1}{2}\sqrt{6} + 1}{\frac{6}{4} - 1} = 4 + (\sqrt{6} - 2).
\end{aligned}$$

Now we see that we have a repetition, and putting everything together, we get the same expansion as before, only this time there is no need for verification.

5. Best approximations

Let x be an irrational number, and $\frac{h_n}{k_n}$ its nth convergent, as defined in the previous section.

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Theorem 12

For all $n \geq 0$ *we have*

$$\left| x - \frac{h_n}{k_n} \right| < \frac{1}{k_n k_{n+1}}.$$

Proof: As before, set $r_n = h_n/k_n$. We saw earlier that

$$|x - r_n| = \frac{1}{k_n(x_{n+1}k_n + k_{n-1})},$$

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and using the fact that $a_{n+1} = \lfloor x_{n+1} \rfloor < x_{n+1}$, we get

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Theorem 13

The convergents get successively closer to x, i.e.,

$$\left| x - \frac{h_n}{k_n} \right| < \left| x - \frac{h_{n-1}}{k_{n-1}} \right|.$$

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Proof is similar in nature to the above.

To put the following main result in perspective, first recall that the rational numbers are dense in the reals, in other words, any real number, rational or irrational, can be arbitrarily closely approximated by rationals.

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In our example, we have the sequence

$$\frac{3}{1}, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \frac{31415}{10000}, \frac{314159}{100000}, \ldots.$$

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In our example, we have the sequence

$$\frac{3}{1}, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \frac{31415}{10000}, \frac{314159}{100000}, \ldots.$$

The following theorem says:
Given a convergent of x, there is no better approximation to x with denominator equal to, or smaller than, that of the convergent. Thus the nth convergents give us a sequence of *best rational approximations*.

Theorem 14

If $\frac{a}{b} \in \mathbb{Q}$, $b > 0$, *and if*

$$\left|x - \frac{a}{b}\right| < \left|x - \frac{h_n}{k_n}\right|$$

for some $n \geq 1$, *then* $b > k_n$.

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This follow from the somewhat stronger statement: If

$$|xb - a| < |xk_n - h_n|$$

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The proof uses similar methods again.

Example: We can obtain numerically

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \ddots}}}}}$$

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and we compute

$$\frac{h_1}{k_1} = 3 + \frac{1}{7} = \frac{22}{7} \simeq 3.142857,$$

which is a better approximation to π than 3.14.

The next two C.F. approximations are

$$\frac{333}{106}, \quad \frac{355}{113}.$$

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and the third one to Adriaan Metius (1571–1635), as good approximations to π.

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Examples of various continued fractions:

$$e = \langle 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots \rangle$$
$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}}$$

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A simple C.F. with a very nice and simple pattern.
But not periodic. Note: e is transcendental.

$$\frac{\pi}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\frac{1}{4} + \ddots}}}}$$

Not simple: Non-integer partial quotients.

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}$$

Not simple: Numerators different from 1.

$$e^z = 1 + \cfrac{z}{1 - \cfrac{z}{2 + \cfrac{z}{1 - \cfrac{z}{3 + \cfrac{z}{1 - \ddots}}}}}$$

Not simple; represents a well-known function. “Analytic theory of continued fractions."