Lecture 2: Periodic Continued Fractions and Pell's Equation

Karl Dilcher

Dalhousie University, Halifax, Canada

December 14, 2018

1. Periodic continued fractions

Here: The second of two main properties of simple C.F.s:
The characterization of quadratic irrationals.

1. Periodic continued fractions

Here: The second of two main properties of simple C.F.s: The characterization of quadratic irrationals.

Definition 1

An infinite simple C.F. $\langle a_0, a_1, a_2, \ldots \rangle$ is *periodic* if there is an $n \in \mathbb{N}$ such that

$$a_j = a_{j+n} \quad \text{for all sufficiently large } j$$

1. Periodic continued fractions

Here: The second of two main properties of simple C.F.s: The characterization of quadratic irrationals.

Definition 1

An infinite simple C.F. $\langle a_0, a_1, a_2, \ldots \rangle$ is *periodic* if there is an $n \in \mathbb{N}$ such that

$$a_j = a_{j+n} \quad \text{for all sufficiently large } j$$

This means: We can write a periodic C.F. as

$$\langle b_0, b_1, \ldots, b_r, a_0, a_1, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{n-1}, \ldots \rangle$$
$$= \langle b_0, b_1, \ldots, b_r, \overline{a_0, a_1, \ldots, a_{n-1}} \rangle$$

1. Periodic continued fractions

Here: The second of two main properties of simple C.F.s: The characterization of quadratic irrationals.

Definition 1

An infinite simple C.F. $\langle a_0, a_1, a_2, \ldots \rangle$ is *periodic* if there is an $n \in \mathbb{N}$ such that

$$a_j = a_{j+n} \quad \text{for all sufficiently large } j$$

This means: We can write a periodic C.F. as

$$\begin{aligned} &\langle b_0, b_1, \ldots, b_r, a_0, a_1, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{n-1}, \ldots \rangle \\ &= \langle b_0, b_1, \ldots, b_r, \overline{a_0, a_1, \ldots, a_{n-1}} \rangle \end{aligned}$$

A periodic C.F. is *purely periodic* if it has the form

$$\langle \overline{a_0, a_1, \ldots, a_{n-1}} \rangle.$$

Example: $\langle\overline{2,3}\rangle = \langle 2,3,2,3,\ldots\rangle.$

Example: $\langle\overline{2,3}\rangle = \langle 2,3,2,3,\dots\rangle$.

If we set $\theta = \langle\overline{2,3}\rangle$, then

$$\theta = 2 + \cfrac{1}{3 + \cfrac{1}{\theta}},$$

Example: $\langle\overline{2,3}\rangle = \langle 2,3,2,3,\ldots\rangle$.

If we set $\theta = \langle\overline{2,3}\rangle$, then

$$\theta = 2 + \cfrac{1}{3 + \cfrac{1}{\theta}},$$

which can be rewritten as a quadratic equation:

$$\begin{aligned} \theta\left(3+\frac{1}{\theta}\right) &= 2\left(3+\frac{1}{\theta}\right), \\ 3\theta^2+\theta &= 6\theta+2+\theta, \\ 3\theta^2-6\theta-2 &= 0. \end{aligned}$$

Example: $\langle \overline{2,3} \rangle = \langle 2,3,2,3,\ldots \rangle$.

If we set $\theta = \langle \overline{2,3} \rangle$, then

$$\theta = 2 + \cfrac{1}{3 + \cfrac{1}{\theta}},$$

which can be rewritten as a quadratic equation:

$$\begin{aligned} \theta\left(3 + \frac{1}{\theta}\right) &= 2\left(3 + \frac{1}{\theta}\right), \\ 3\theta^2 + \theta &= 6\theta + 2 + \theta, \\ 3\theta^2 - 6\theta - 2 &= 0. \end{aligned}$$

This has the solution

$$\theta = \frac{3 + \sqrt{15}}{3}.$$

Example: $\langle\overline{2,3}\rangle = \langle 2,3,2,3,\ldots\rangle$.

If we set $\theta = \langle\overline{2,3}\rangle$, then

$$\theta = 2 + \cfrac{1}{3 + \cfrac{1}{\theta}},$$

which can be rewritten as a quadratic equation:

$$\begin{aligned} \theta\left(3+\frac{1}{\theta}\right) &= 2\left(3+\frac{1}{\theta}\right), \\ 3\theta^2+\theta &= 6\theta+2+\theta, \\ 3\theta^2-6\theta-2 &= 0. \end{aligned}$$

This has the solution

$$\theta = \frac{3+\sqrt{15}}{3}.$$

Any periodic C.F. can be evaluated in this way.

Example: $\langle\overline{2,3}\rangle = \langle 2,3,2,3,\ldots\rangle$.

If we set $\theta = \langle\overline{2,3}\rangle$, then

$$\theta = 2 + \cfrac{1}{3 + \cfrac{1}{\theta}},$$

which can be rewritten as a quadratic equation:

$$\begin{aligned} \theta\left(3+\frac{1}{\theta}\right) &= 2\left(3+\frac{1}{\theta}\right), \\ 3\theta^2+\theta &= 6\theta+2+\theta, \\ 3\theta^2-6\theta-2 &= 0. \end{aligned}$$

This has the solution

$$\theta = \frac{3+\sqrt{15}}{3}.$$

Any periodic C.F. can be evaluated in this way.
Do we always get a quadratic irrational?

Theorem 2 (Lagrange)

Any periodic simple C.F. is a quadratic irrational, and conversely.

Theorem 2 (Lagrange)

Any periodic simple C.F. is a quadratic irrational, and conversely.

Proof. The two directions have to be proven separately.

Theorem 2 (Lagrange)

Any periodic simple C.F. is a quadratic irrational, and conversely.

Proof. The two directions have to be proven separately.

"$\Rightarrow$": Let x be the value of the full C.F., and θ its periodic part:

$$\begin{aligned} \theta &= \langle \overline{a_0, a_1, \ldots, a_{n-1}} \rangle \\ &= \langle a_0, a_1, \ldots, a_{n-1}, \theta \rangle. \end{aligned}$$

Theorem 2 (Lagrange)

Any periodic simple C.F. is a quadratic irrational, and conversely.

Proof. The two directions have to be proven separately.

"$\Rightarrow$": Let x be the value of the full C.F., and θ its periodic part:

$$\begin{aligned}\theta &= \langle\overline{a_0, a_1, \dots, a_{n-1}}\rangle \\ &= \langle a_0, a_1, \dots, a_{n-1}, \theta\rangle.\end{aligned}$$

From the proof of Theorem 1.11 we have

$$\theta = \frac{\theta h_{n-1} + h_{n-2}}{\theta k_{n-1} + k_{n-2}};$$

Theorem 2 (Lagrange)

Any periodic simple C.F. is a quadratic irrational, and conversely.

Proof. The two directions have to be proven separately.

"⇒": Let x be the value of the full C.F., and θ its periodic part:

$$\begin{aligned}\theta &= \langle\overline{a_0, a_1, \ldots, a_{n-1}}\rangle \\ &= \langle a_0, a_1, \ldots, a_{n-1}, \theta\rangle.\end{aligned}$$

From the proof of Theorem 1.11 we have

$$\theta = \frac{\theta h_{n-1} + h_{n-2}}{\theta k_{n-1} + k_{n-2}};$$

Clearly a quadratic equation in θ. Hence θ is either rational or a quadratic irrational.

Theorem 2 (Lagrange)

Any periodic simple C.F. is a quadratic irrational, and conversely.

Proof. The two directions have to be proven separately.

"$\Rightarrow$": Let x be the value of the full C.F., and θ its periodic part:

$$\begin{aligned}\theta &= \langle\overline{a_0, a_1, \ldots, a_{n-1}}\rangle \\ &= \langle a_0, a_1, \ldots, a_{n-1}, \theta\rangle.\end{aligned}$$

From the proof of Theorem 1.11 we have

$$\theta = \frac{\theta h_{n-1} + h_{n-2}}{\theta k_{n-1} + k_{n-2}};$$

Clearly a quadratic equation in θ. Hence θ is either rational or a quadratic irrational.

However, by Theorem 1.8 it cannot be rational since we have an infinite C.F.

Now consider the full C.F.

$$x = \langle b_0, b_1, \ldots, b_r, \theta \rangle = \frac{\theta m + m'}{\theta q + q'},$$

where m'/q', m/q are the last two convergents to $\langle b_0, b_1, \ldots, b_r \rangle$.

Now consider the full C.F.

$$x = \langle b_0, b_1, \ldots, b_r, \theta \rangle = \frac{\theta m + m'}{\theta q + q'},$$

where m'/q', m/q are the last two convergents to $\langle b_0, b_1, \ldots, b_r \rangle$.

But we know that $\theta = (a + \sqrt{b})/c$, so x must be of a similar form;
also, $x \notin \mathbb{Q}$.

Now consider the full C.F.

$$x = \langle b_0, b_1, \ldots, b_r, \theta \rangle = \frac{\theta m + m'}{\theta q + q'},$$

where m'/q', m/q are the last two convergents to $\langle b_0, b_1, \ldots, b_r \rangle$.

But we know that $\theta = (a + \sqrt{b})/c$, so x must be of a similar form;
also, $x \notin \mathbb{Q}$.

"$\Leftarrow$": We skip this; it is rather technical.
Please see notes, or the book by Niven, Zuckerman & Montgomery [NZM]. ■

2. Some special cases

The two results in this section are required for the following section.

Theorem 3

The C.F. expansion of a real quadratic irrational number x is purely periodic if and only if $x > 1$ and $-1 < x' < 0$, where x' is the conjugate of x,

For a proof, see [NZM].

For the next result, we define recursively

$$\begin{aligned} m_{i+1} &= a_i q_i - m_i, \quad q_{i+1} \frac{d - m_{i+1}^2}{q_i}, \qquad (1) \\ x_i &= \frac{m_i + \sqrt{d}}{q_i}, \qquad a_i = \lfloor x_i \rfloor. \end{aligned}$$

For the next result, we define recursively

$$m_{i+1} = a_i q_i - m_i, \quad q_{i+1} \frac{d - m_{i+1}^2}{q_i}, \tag{1}$$

$$x_i = \frac{m_i + \sqrt{d}}{q_i}, \qquad a_i = \lfloor x_i \rfloor.$$

Theorem 4

Let $d > 1$ be an integer, not a perfect square. Then
(1) *the simple C.F. expansion of $\sqrt{d}$ has the form*

$$\sqrt{d} = \langle a_0, \overline{a_1, a_2, \ldots, a_{r-1}, 2a_0} \rangle, \qquad a_0 = \lfloor \sqrt{d} \rfloor.$$

For the next result, we define recursively

$$\begin{aligned} m_{i+1} &= a_i q_i - m_i, \quad q_{i+1} \frac{d - m_{i+1}^2}{q_i}, \qquad (1) \\ x_i &= \frac{m_i + \sqrt{d}}{q_i}, \qquad a_i = \lfloor x_i \rfloor. \end{aligned}$$

Theorem 4

Let $d > 1$ be an integer, not a perfect square. Then
(1) *the simple C.F. expansion of $\sqrt{d}$ has the form*

$$\sqrt{d} = \langle a_0, \overline{a_1, a_2, \ldots, a_{r-1}, 2a_0} \rangle, \qquad a_0 = \lfloor \sqrt{d} \rfloor.$$

(2) *With $x_0 = \sqrt{d}, q_0 = 1, m_0 = 0$ in* (1) *we have*
$q_i = 1$ iff $r \mid i$, while $q_i = -1$ holds for no i.

For the next result, we define recursively

$$
\begin{aligned}
m_{i+1} &= a_i q_i - m_i, \quad q_{i+1} \frac{d - m_{i+1}^2}{q_i}, \qquad (1) \\
x_i &= \frac{m_i + \sqrt{d}}{q_i}, \qquad a_i = \lfloor x_i \rfloor.
\end{aligned}
$$

Theorem 4

Let $d > 1$ be an integer, not a perfect square. Then
(1) *the simple C.F. expansion of $\sqrt{d}$ has the form*

$$\sqrt{d} = \langle a_0, \overline{a_1, a_2, \ldots, a_{r-1}, 2a_0} \rangle, \qquad a_0 = \lfloor \sqrt{d} \rfloor.$$

(2) *With $x_0 = \sqrt{d}, q_0 = 1, m_0 = 0$ in* (1) *we have*
$q_i = 1$ iff $r \mid i$, while $q_i = -1$ holds for no i.

The proof is once again technical, and we skip it; see the notes.

3. Pell's equation

The equation

$$x^2 - d\,y^2 = N, \qquad d, N, \in \mathbb{Z}, \tag{2}$$

with integer unknowns x and y, is usually called *Pell's equation*.

3. Pell's equation

The equation

$$x^2 - d\,y^2 = N, \qquad d, N, \in \mathbb{Z}, \tag{2}$$

with integer unknowns x and y, is usually called *Pell's equation*.

Some remarks:

– The original case is $N = 1$.

The equation

$$x^2 - d\,y^2 = N, \qquad d, N, \in \mathbb{Z}, \tag{2}$$

with integer unknowns x and y, is usually called *Pell's equation*.

Some remarks:

– The original case is $N = 1$.

– Important applications in algebraic number theory.

3. Pell's equation

The equation

$$x^2 - d\,y^2 = N, \qquad d, N, \in \mathbb{Z}, \tag{2}$$

with integer unknowns x and y, is usually called *Pell's equation*.

Some remarks:

– The original case is $N = 1$.

– Important applications in algebraic number theory.

– If d is negative, then (2) can only have finitely many solutions.

3. Pell's equation

The equation

$$x^2 - d\,y^2 = N, \qquad d, N, \in \mathbb{Z}, \tag{2}$$

with integer unknowns x and y, is usually called *Pell's equation*.

Some remarks:

– The original case is $N = 1$.

– Important applications in algebraic number theory.

– If d is negative, then (2) can only have finitely many solutions.

– If d is a perfect square, say $d = a^2$, then (2) reduces to

$$(x - ay)(x + ay) = N,$$

which again can only have finitely many solutions.
Therefore we assume that $d > 1$, not a perfect square.

We expand $\sqrt{d}$ into a C.F. as in Theorem 4, with convergents h_n/k_n, and with q_n defined by (1),
with $x_0 = \sqrt{d}$, $q_0 = 1$, $m_0 = 0$.

We expand $\sqrt{d}$ into a C.F. as in Theorem 4, with convergents h_n/k_n, and with q_n defined by (1),
with $x_0 = \sqrt{d}$, $q_0 = 1$, $m_0 = 0$.

Theorem 5

If d is a positive integer not a perfect square, then for all integers $n \geq -1$ we have

$$h_n^2 - d \cdot k_n^2 = (-1)^{n-1} q_{n+1}.$$

Sketch of proof:

From Lecture 1 and (1) we have

$$\sqrt{d} = x_0 = \frac{x_{n+1}h_n + h_{n+1}}{x_{n+1}k_n + k_{n+1}} = \frac{(m_{n+1} + \sqrt{d})h_n + q_{n+1}h_{n-1}}{(m_{n+1} + \sqrt{d})k_n + q_{n+1}k_{n-1}}.$$

Sketch of proof:
From Lecture 1 and (1) we have

$$\sqrt{d} = x_0 = \frac{x_{n+1}h_n + h_{n+1}}{x_{n+1}k_n + k_{n+1}} = \frac{(m_{n+1} + \sqrt{d})h_n + q_{n+1}h_{n-1}}{(m_{n+1} + \sqrt{d})k_n + q_{n+1}k_{n-1}}.$$

– Simplify this equation,
– separate into rational and purely irrational parts.
– Each part must be 0, so we get two equations;
– eliminate m_{n+1} from them.
The final result is

$$h_n^2 - d \cdot k_n^2 = (h_n k_{n-1} - h_{n-1}k_n)q_{n+1} = (-1)^{n-1}q_{n+1},$$

where we have used a result from Lecture 1 in the last step.

Corollary 6

Let r be the period length of the expansion of $\sqrt{d}$ in Theorem 4. Then for $n \geq 0$ we have

$$h_{nr+1}^2 - d \cdot k_{nr+1}^2 = (-1)^{nr} q_{nr} = (-1)^{nr}. \tag{3}$$

Corollary 6

Let r be the period length of the expansion of $\sqrt{d}$ in Theorem 4. Then for $n \geq 0$ we have

$$h_{nr+1}^2 - d \cdot k_{nr+1}^2 = (-1)^{nr} q_{nr} = (-1)^{nr}. \tag{3}$$

Proof: Immediate from Theorem 5 with n replaced by $nr - 1$, and then using Theorem 4(2). ∎

Corollary 6

Let r be the period length of the expansion of $\sqrt{d}$ in Theorem 4. Then for $n \geq 0$ we have

$$h_{nr+1}^2 - d \cdot k_{nr+1}^2 = (-1)^{nr} q_{nr} = (-1)^{nr}. \tag{3}$$

Proof: Immediate from Theorem 5 with n replaced by $nr - 1$, and then using Theorem 4(2). ■

We can now see: (3) gives us infinitely many solutions of the equations

$$x^2 - d \cdot y^2 = 1 \quad \text{and} \quad x^2 - d \cdot y^2 = -1.$$

Corollary 6

Let r be the period length of the expansion of $\sqrt{d}$ in Theorem 4. Then for $n \geq 0$ we have

$$h_{nr+1}^2 - d \cdot k_{nr+1}^2 = (-1)^{nr} q_{nr} = (-1)^{nr}. \qquad (3)$$

Proof: Immediate from Theorem 5 with n replaced by $nr - 1$, and then using Theorem 4(2). ∎

We can now see: (3) gives us infinitely many solutions of the equations

$$x^2 - d \cdot y^2 = 1 \quad \text{and} \quad x^2 - d \cdot y^2 = -1.$$

Are there any other solutions?

Corollary 6

Let r be the period length of the expansion of $\sqrt{d}$ in Theorem 4. Then for $n \geq 0$ we have

$$h_{nr+1}^2 - d \cdot k_{nr+1}^2 = (-1)^{nr} q_{nr} = (-1)^{nr}. \tag{3}$$

Proof: Immediate from Theorem 5 with n replaced by $nr - 1$, and then using Theorem 4(2). ■

We can now see: (3) gives us infinitely many solutions of the equations

$$x^2 - d \cdot y^2 = 1 \quad \text{and} \quad x^2 - d \cdot y^2 = -1.$$

Are there any other solutions?
We'll see: *Every* solution of these equations can be obtained from the C.F. expansion of $\sqrt{d}$.

Note: sufficient to consider the positive solutions $x > 0$, $y > 0$.

First we need two other result that are of independent interest.

Note: sufficient to consider the positive solutions $x > 0$, $y > 0$.

First we need two other result that are of independent interest.

Theorem 7

Let x be an irrational number. If there is an $r/s \in \mathbb{Q}$ with $(r, s) = 1$ and $s \geq 1$ such that

$$\left| x - \frac{r}{s} \right| < \frac{1}{2s^2},$$

then r/s is one of the convergents of the simple C.F. expansion of x.

Note: sufficient to consider the positive solutions $x > 0$, $y > 0$.

First we need two other result that are of independent interest.

Theorem 7

Let x be an irrational number. If there is an $r/s \in \mathbb{Q}$ with $(r, s) = 1$ and $s \geq 1$ such that

$$\left| x - \frac{r}{s} \right| < \frac{1}{2s^2},$$

then r/s is one of the convergents of the simple C.F. expansion of x.

Proof: Follows from the best approximation theorem.

Theorem 8

Let $x > 1$ be a real number. Then the nth convergent of $1/x$ is the reciprocal of the $(n-1)$st convergent of x.

Theorem 8

Let $x > 1$ be a real number. Then the nth convergent of $1/x$ is the reciprocal of the $(n-1)$st convergent of x.

Proof: Follows from induction, with results of Lecture 1.

Theorem 8

Let $x > 1$ be a real number. Then the nth convergent of $1/x$ is the reciprocal of the $(n-1)$st convergent of x.

Proof: Follows from induction, with results of Lecture 1.

Theorem 9

Let d be a positive integer not a perfect square, and let the convergents of the simple continued fraction expansion of $\sqrt{d}$ be h_n/k_n. Suppose the integer N satisfies $|N| < d$. Then any positive solution $x = s$, $y = t$ of

$$x^2 - d \cdot y^2 = N \quad \textit{with} \quad (s, t) = 1$$

satisfies $s = h_n$, $t = k_n$ for some positive integer n.

Theorem 8

Let $x > 1$ be a real number. Then the nth convergent of $1/x$ is the reciprocal of the $(n-1)$st convergent of x.

Proof: Follows from induction, with results of Lecture 1.

Theorem 9

Let d be a positive integer not a perfect square, and let the convergents of the simple continued fraction expansion of $\sqrt{d}$ be h_n/k_n. Suppose the integer N satisfies $|N| < d$. Then any positive solution $x = s$, $y = t$ of

$$x^2 - d \cdot y^2 = N \quad \text{with} \quad (s, t) = 1$$

satisfies $s = h_n$, $t = k_n$ for some positive integer n.

Proof uses Theorems 7 and 8.

Combining Theorems 4, 5, and 9, we get the following result.

Theorem 10

Let $d > 1$ *be an integer not a perfect square, let* h_n/k_n *be the convergents of the C.F. expansion of* $\sqrt{d}$*, and let* r *be the period of the expansion of* $\sqrt{d}$ *as given in Theorem 4.*

Combining Theorems 4, 5, and 9, we get the following result.

Theorem 10

Let $d > 1$ be an integer not a perfect square, let h_n/k_n be the convergents of the C.F. expansion of $\sqrt{d}$, and let r be the period of the expansion of $\sqrt{d}$ as given in Theorem 4.Then

(1) *all positive solutions of $x^2 - d\,y^2 = \pm 1$ can be found among $x = h_n$, $y = k_n$;*

Combining Theorems 4, 5, and 9, we get the following result.

Theorem 10

Let $d > 1$ be an integer not a perfect square, let h_n/k_n be the convergents of the C.F. expansion of $\sqrt{d}$, and let r be the period of the expansion of $\sqrt{d}$ as given in Theorem 4.Then

(1) *all positive solutions of $x^2 - d\,y^2 = \pm 1$ can be found among $x = h_n$, $y = k_n$;*

(2) *if r is even, then $x^2 - d\,y^2 = -1$ has no solution, and all positive solutions of $x^2 - d\,y^2 = 1$ are given by $x = h_{nr-1}$, $y = k_{nr-1}$ for $n = 1, 2, 3, \ldots$;*

Combining Theorems 4, 5, and 9, we get the following result.

Theorem 10

Let $d > 1$ be an integer not a perfect square, let h_n/k_n be the convergents of the C.F. expansion of $\sqrt{d}$, and let r be the period of the expansion of $\sqrt{d}$ as given in Theorem 4.Then

1. *all positive solutions of $x^2 - d\,y^2 = \pm 1$ can be found among $x = h_n$, $y = k_n$;*
2. *if r is even, then $x^2 - d\,y^2 = -1$ has no solution, and all positive solutions of $x^2 - d\,y^2 = 1$ are given by $x = h_{nr-1}$, $y = k_{nr-1}$ for $n = 1, 2, 3, \ldots$;*
3. *if r is odd, then $x = h_{nr-1}$, $y = k_{nr-1}$ give all positive solutions of $x^2 - d\,y^2 = -1$ for $n = 1, 3, 5, \ldots$, and all positive solutions of $x^2 - d\,y^2 = 1$ for $n = 2, 4, 6, \ldots$.*

Example: Consider

$$x^2 - 73y^2 = -1.$$

Example: Consider

$$x^2 - 73y^2 = -1.$$

We first find the C.F. expansion

$$\sqrt{73} = \langle 8, \overline{1, 1, 5, 5, 1, 1, 16} \rangle.$$

Example: Consider

$$x^2 - 73y^2 = -1.$$

We first find the C.F. expansion

$$\sqrt{73} = \langle 8, \overline{1, 1, 5, 5, 1, 1, 16} \rangle.$$

Hence $r = 7$, and by Theorem 10 the equation has solutions, and the least positive solution is given by h_6/k_6.

Example: Consider

$$x^2 - 73y^2 = -1.$$

We first find the C.F. expansion

$$\sqrt{73} = \langle 8, \overline{1, 1, 5, 5, 1, 1, 16} \rangle.$$

Hence $r = 7$, and by Theorem 10 the equation has solutions, and the least positive solution is given by h_6/k_6.

We use the recurrences from Lecture 1 to obtain successive h_j/k_j, starting with h_0/k_0:

$$\frac{8}{1}, \quad \frac{9}{1}, \quad \frac{17}{2}, \quad \frac{94}{11}, \quad \frac{487}{57}, \quad \frac{561}{68}, \quad \frac{1068}{125}.$$

Example: Consider

$$x^2 - 73y^2 = -1.$$

We first find the C.F. expansion

$$\sqrt{73} = \langle 8, \overline{1, 1, 5, 5, 1, 1, 16} \rangle.$$

Hence $r = 7$, and by Theorem 10 the equation has solutions, and the least positive solution is given by h_6/k_6.

We use the recurrences from Lecture 1 to obtain successive h_j/k_j, starting with h_0/k_0:

$$\frac{8}{1}, \quad \frac{9}{1}, \quad \frac{17}{2}, \quad \frac{94}{11}, \quad \frac{487}{57}, \quad \frac{561}{68}, \quad \frac{1068}{125}.$$

Therefore $x = 1068$, $y = 125$ is the smallest positive solution of the given equation.

In a similar way we can find the smallest positive solution of the related equation

$$x^2 - 73y^2 = 1.$$

In a similar way we can find the smallest positive solution of the related equation

$$x^2 - 73y^2 = 1.$$

In this case we need to compute recursively the fraction h_{13}/k_{13}, which leads to the solution $x = 2\,281\,249$, $y = 267\,000$.

In a similar way we can find the smallest positive solution of the related equation

$$x^2 - 73y^2 = 1.$$

In this case we need to compute recursively the fraction h_{13}/k_{13}, which leads to the solution $x = 2\,281\,249$, $y = 267\,000$.

All other solutions can also be found in this way.