# Lecture 3:
# Chebyshev's prime number theorem

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## 1. Introduction

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**Theorem 2 (Euclid)**

*There are infinitely many primes.*

**Proof:** Suppose there were only finitely mny primes, namely $p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_r$.

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$p_r$: remainder 1 upon division.

Hence any prime divisor of $n$ must be distinct from the $p_1, p_2, \ldots, p_r$, but this is a contradiction.

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Second, there are the **twin primes**, namely pairs $p, p + 2$ of primes, such as

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Not known: The **twin-prime conjecture**, one of the most famous unsolved problems in number theory.

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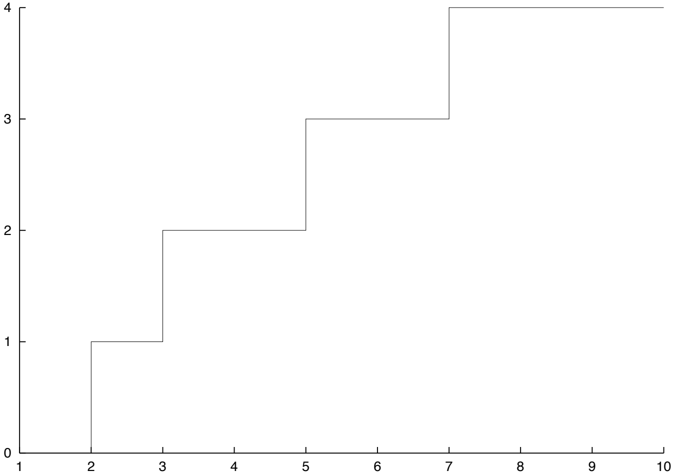

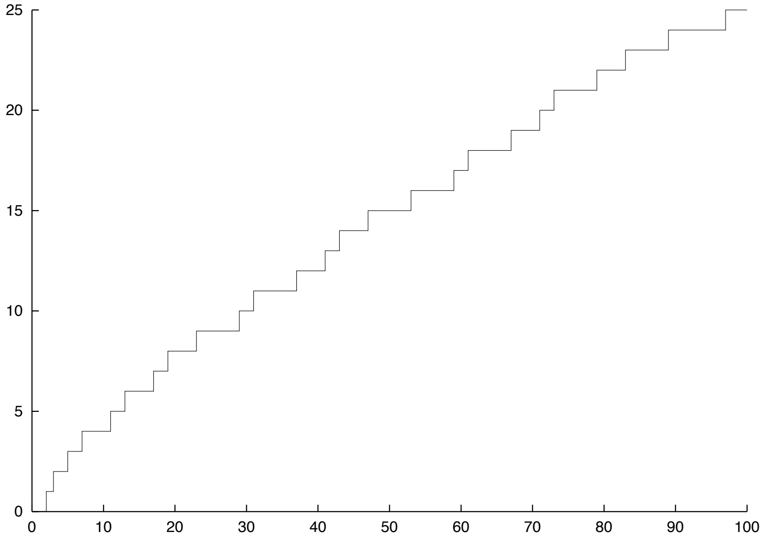
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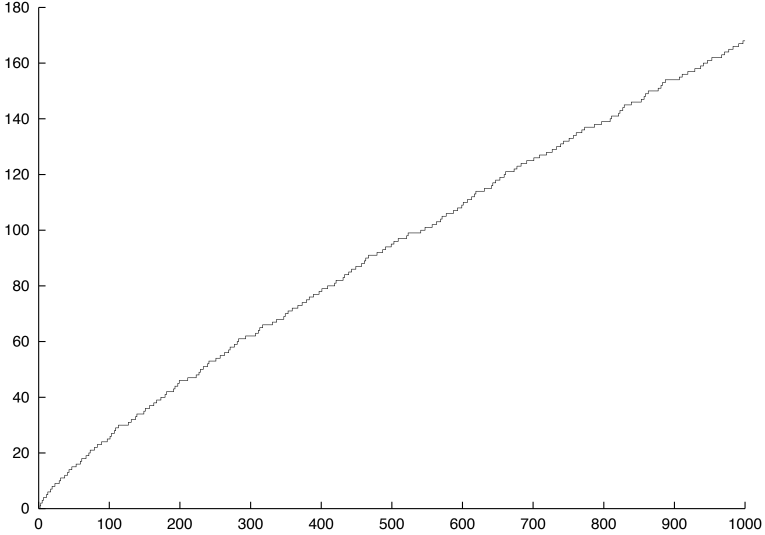
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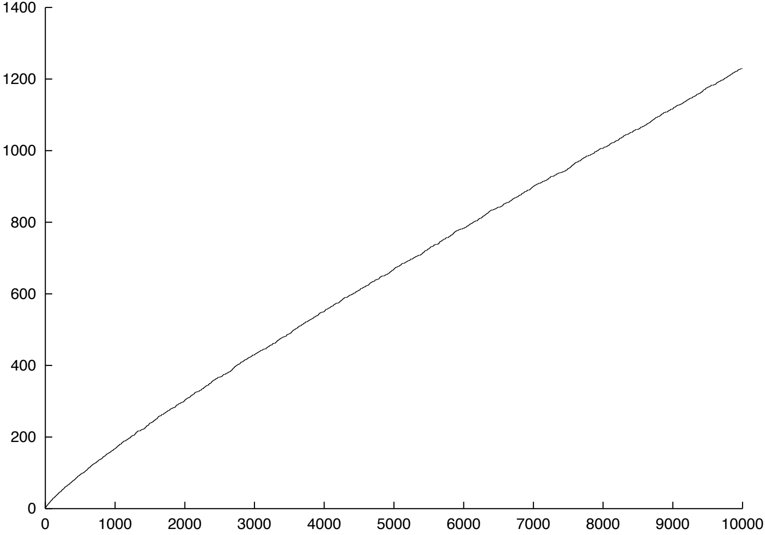
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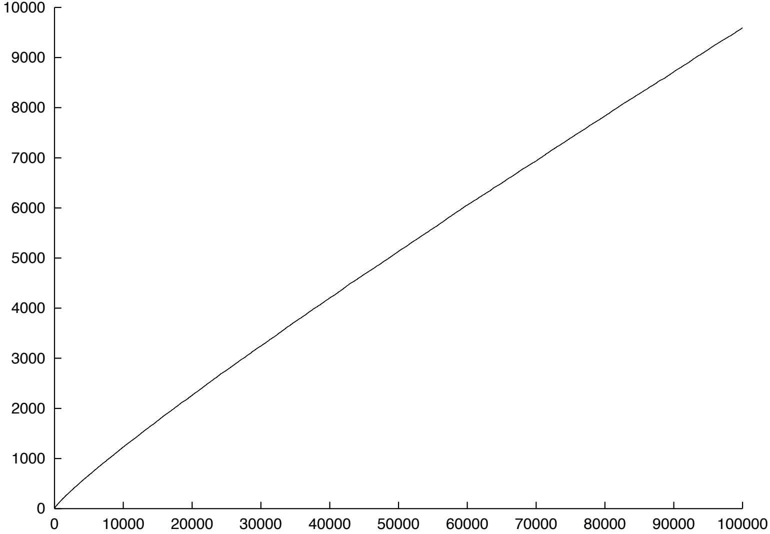
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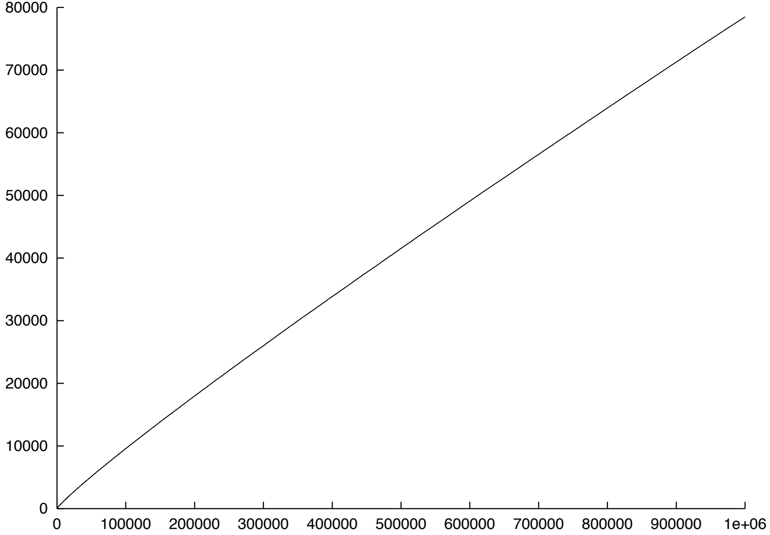
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It was proved by Jacques Hadamard (1865–1963) and (independently) Charles de la Vallée Poussin (1866–1962) in 1896 that

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which is a short form for writing

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1.$$

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To prove this theorem, we first state another one:

## Theorem 4

*For all* $n \geq 2$,

$$\frac{1}{8} \leq \pi(n)\frac{H(n)}{n} < 6,$$

*where*

$$H(n) = \sum_{j=2}^{n} \frac{1}{j}.$$

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*where*

$$H(n) = \sum_{j=2}^{n} \frac{1}{j}.$$

In other words: $\pi(n)$ is of the same order of magnitude as the reciprocal of the average of $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n})$.

**Proof of Theorem 3** For $n \geq 2$ we have

$$\log \frac{n}{2} = \int_2^n \frac{dt}{t} < \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{dt}{t} = \log n.$$

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Hence for all $n \geq 2$,

$$\frac{1}{2} \log n \leq H(n) \leq \log n,$$

and therefore Theorem 3 follows from Theorem 4.

**Remarks:** (1) The numbers $H(n)$ are closely related to the **harmonic numbers**

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After the numbers $\pi$ and $e$ it is probably the most important special constant in mathematics, and it is also closely related to the **Gamma function** $\Gamma(x)$.

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For $n > 9$ we clearly have $\pi(n) \le n/2$.
For the smaller cases we count the primes:

$$\pi(2) = 1 = 2^0, \quad \pi(4) = 2 = 2^1, \quad \pi(8) = 4 = 2^2.$$

## Lemma 6

$$\frac{1}{2}l \leq H(2^l) \leq l.$$

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**Proof:** Group the terms of the sum $H(2^l)$ in 2 different ways:

$$\begin{aligned}
H(2^l) &= \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&\quad + \left(\frac{1}{2^{l-1}+1} + \cdots + \frac{1}{2^l}\right) \\
&\ge \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&\quad + \left(\frac{1}{2^l} + \cdots + \frac{1}{2^l}\right) \\
&= \frac{1}{2}l.
\end{aligned}$$

On the other hand,

$$\begin{aligned} H(2^l) &= \left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\cdots+\frac{1}{2^l} \\ &\le \left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\cdots \\ &\quad +\left(\frac{1}{2^{l-1}}+\cdots+\frac{1}{2^{l-1}}+\frac{1}{2^l}\right) \\ &\le l. \end{aligned}$$

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$$1,\ 2,\ 3,\ \ldots, n-1,\ n \quad \text{of} \quad n!,$$

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– every $p^2$th number (namely $p, 2p, \ldots$) is divisible by another $p$; there are $\lfloor n/p^2 \rfloor$ of them;
– and so on, so that the power of $p$ in $n!$ is

$$\lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \cdots$$

## Lemma 8

*The power of a prime p in* $\binom{N}{n}$ *is*

$$\sum_{m\geq 1}\left(\left\lfloor\frac{N}{p^m}\right\rfloor-\left\lfloor\frac{n}{p^m}\right\rfloor-\left\lfloor\frac{N-n}{p^m}\right\rfloor\right).$$

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**Proof:** We use the fact that

$$\binom{N}{n}=\frac{N!}{n!(N-n)!}$$

and apply Lemma 7.

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(i) We begin with the following assertion:

$$\prod_{n<p\leq 2n} p \,\Big|\, \binom{2n}{n} \qquad \text{and} \qquad \binom{2n}{n} \,\Big|\, \prod_{p^r\leq 2n<p^{r+1}} p^r. \tag{1}$$

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This proves the left part of (1).

On the other hand, by Lemma 8 the power of $p$ in $\binom{2n}{n}$ is

$$\sum_{m=1}^{r} \underbrace{\left( \left\lfloor \frac{2n}{p^n} \right\rfloor - 2 \left\lfloor \frac{n}{p^n} \right\rfloor \right)}_{\leq 1} \leq r.$$

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(ii) Changing divisibility in (1) into inequalities, we get for $n \geq 1$,

$$n^{\pi(2n)-\pi(n)} < \prod_{n<p\leq 2n} p \leq \binom{2n}{n} \leq \prod_{p^r \leq 2n < p^{r+1}} p^r \leq (2n)^{\pi(2n)}. \quad (2)$$

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(ii) Changing divisibility in (1) into inequalities, we get for $n \geq 1$,

$$n^{\pi(2n)-\pi(n)} < \prod_{n<p\leq 2n} p \leq \binom{2n}{n} \leq \prod_{p^r\leq 2n<p^{r+1}} p^r \leq (2n)^{\pi(2n)}. \quad (2)$$

Left inequality follows from: $\pi(2n) - \pi(n)$ primes between $n$ and $2n$, and the smallest one is at least $n$.

On the other hand, by Lemma 8 the power of $p$ in $\binom{2n}{n}$ is

$$\sum_{m=1}^{r} \underbrace{\left( \left\lfloor \frac{2n}{p^n} \right\rfloor - 2 \left\lfloor \frac{n}{p^n} \right\rfloor \right)}_{\leq 1} \leq r.$$

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Right inequality follows from: Number of $p$ in the preceding product is $\pi(2n)$, while each factor $p^r$ is at most $2n$.

Now

$$
\begin{aligned}
\binom{2n}{n} &= \frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1} \\
&= 2\left(2+\frac{1}{n-1}\right)\cdots\left(2+\frac{j}{n-j}\right)\cdots\left(2+\frac{n-1}{1}\right) \\
&\geq 2^n,
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Hence, with (2) we get for $n \geq 1$,

$$n^{\pi(2n)-\pi(n)} < 2^{2n}, \qquad 2^{2n} \leq (2n)^{\pi(2n)}.$$

Now let $n = 2^k$, $k = 0, 1, 2, \ldots$.

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or

$$k(\pi(2^{k+1}) - \pi(2^k)) < 2^{k+1}, \qquad 2^k \leq (k+1)\pi(2^{k+1}). \qquad (3)$$

Now

$$(k+1)\pi(2^{k+1}) - k\pi(2^k) < 2^{k+1} + \pi(2^{k+1}) \leq 3 \cdot 2^k,$$

where the first inequality follows from (3), and the second one from Lemma 5.

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Replace $k$ successively by $0, 1, \ldots, k$ and add.
On the left we have a “telescoping sum" so that

$$(k+1)\pi(2^{k+1}) < 3\left(2^0 + 2^1 + \cdots + 2^k\right) < 3 \cdot 2^{k+1}.$$

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Using this and (3), we find

$$\frac{1}{2} \cdot \frac{2^{k+1}}{k+1} \le \pi(2^{k+1}) < 3\frac{2^{k+1}}{k+1}.$$

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Now let $n \in \mathbb{N}$, $n > 1$, be such that

$$2^{k+1} \le n < 2^{k+2}.$$

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Then

$$\pi(n) \leq \pi(2^{k+2}) < 3\frac{2^{k+2}}{k+2} \leq 6\frac{2^{k+1}}{H(2^{k+2})} \leq 6\frac{n}{H(n)},$$

where we have used Lemma 6 in the third inequality.

On the other hand,

$$\pi(n) \geq \pi(2^{k+1}) \geq \frac{1}{2} \cdot \frac{2^{k+1}}{k+1} = \frac{1}{8} \cdot \frac{2^{k+2}}{\frac{1}{2}(k+1)}$$
$$\geq \frac{1}{8} \cdot \frac{2^{k+2}}{H(2^{k+1})} \geq \frac{1}{8} \cdot \frac{n}{H(n)},$$

where we have used Lemma 6 for the second-last inequality.

On the other hand,

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Putting the last two strings of inequalities together, we finally obtain the statement of Theorem 4.

Consequence of Chebyshev's theorem:

Proof of a famous and at that time unsolved problem, namely

**Bertrand's Postulate:** (J. L. F. Bertrand, 1845)

*If* $x \in \mathbb{R}$*,* $x > 1$*, then there is at least one prime in the open interval* $(x, 2x)$*.*

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Chebyshev said,
And I'll say it again,
There's always a prime
Between n and 2n.