Ramanujan's theta functions and series for $1/\pi$

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A theta function is a series of the form

$$\sum_{n=-\infty}^{\infty} q^{n^2} x^n, \quad \text{where} \quad |q| < 1 \quad \text{and} \quad 0 < |x| < \infty,$$

or a generalization or specialization of such a series. Theta functions occur famously in applied mathematics and physics as solutions of partial differential equations. Just as importantly, theta functions play a fundamental role in analytic number theory where they are related to the Riemann zeta function, modular forms and elliptic functions.

This lecture course will survey the main properties of theta functions that have applications in number theory. We take inspiration from Ramanujan and study a large number of explicit examples. Along the way we will encounter some of the most beautiful formulas in mathematics and use them as motivation to initiate the study of modular forms.

Some recent developments will be described, including Ramanujan's extensions of Jacobi's inversion formula to alternative bases. If time permits, we will discuss the corresponding results for higher levels where sequences such as the Apéry numbers arise.

The lectures will culminate with an analysis of a class of series for $1/\pi$ first studied by Ramanujan and a discussion of some open problems.

Reference: "Ramanujan's Theta Functions", S. Cooper, Springer, 2017.