Transformations

Shaun Cooper

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$$z(\theta) = \frac{1}{2}\cot\frac{\theta}{2} - i\sum_{n=1}^{\infty}\left(\frac{e^{i\theta}q^n}{1-e^{i\theta}q^n} - \frac{e^{-i\theta}q^n}{1-e^{-i\theta}q^n}\right)$$

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A q-analogue of the cotangent

In fact, when $q \to 0$, $z(\theta) = \frac{1}{2}\cot\frac{\theta}{2}$. (What happens to τ?)

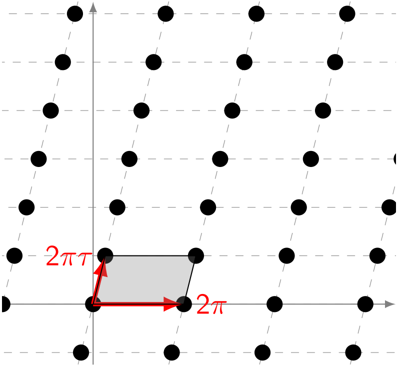
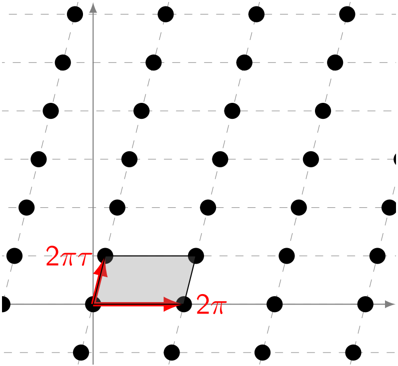


Figure: The lattice Λ, generated by 2π and $2\pi\tau$.



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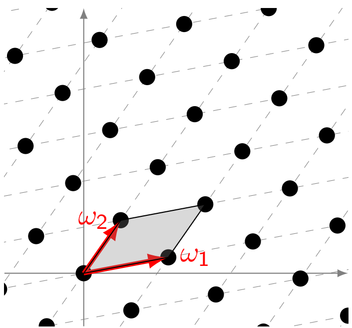
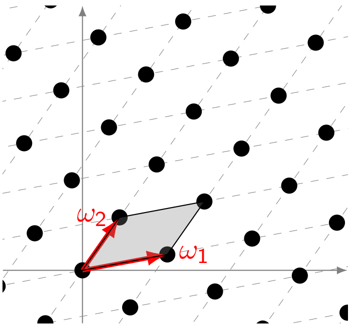


Figure: The lattice generated by ω_1 and ω_2. $\mathrm{Im}(\omega_2/\omega_1) > 0$.
$\Lambda(\omega_1, \omega_2) = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}$.



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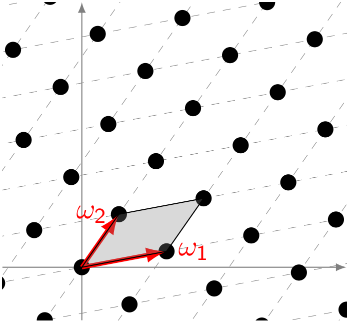
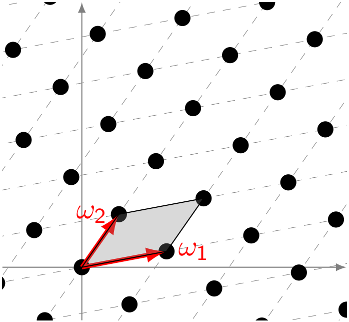
Create a meromorphic function with simple poles at each point in $\Lambda(\omega_1, \omega_2)$ with residue 1 at each pole.



Let $Z(\theta, \omega_1, \omega_2) = \dfrac{2\pi}{\omega_1}\, z\left(\dfrac{2\pi\theta}{\omega_1}, \dfrac{\omega_2}{\omega_1}\right).$ Assume $\mathrm{Im}\left(\dfrac{\omega_2}{\omega_1}\right) > 0.$

Poles at each point in $\Lambda(\omega_1, \omega_2)$ with residue 1



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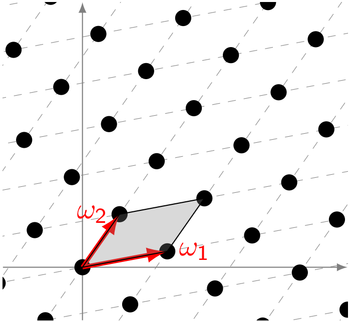

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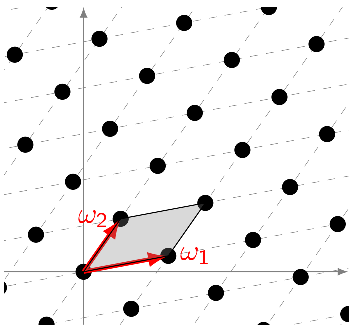

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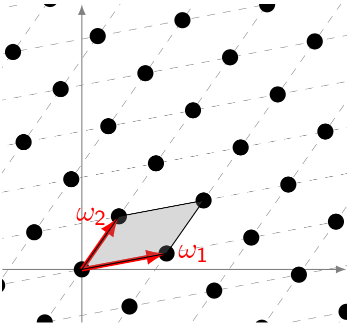

Suppose a, b, c, $d \in \mathbb{Z}$ and $ad - bc = 1$.

Let ω_3 and ω_4 be defined by $\begin{bmatrix} \omega_4 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$.

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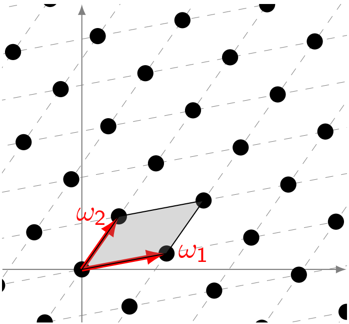

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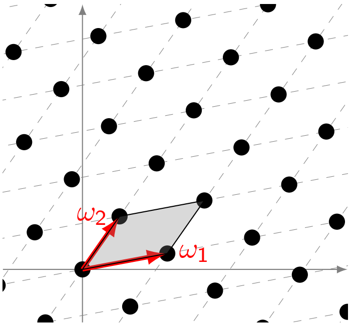

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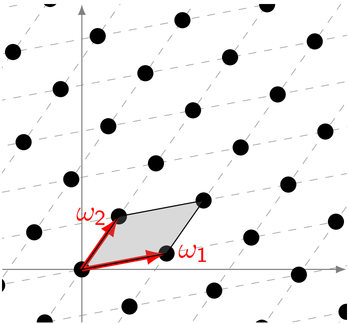

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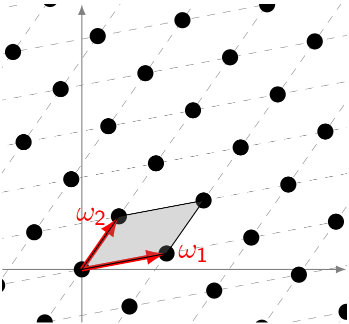

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Theorem

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The reverse inclusion is obtained in the same way, using

$$\begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} \omega_4 \\ \omega_3 \end{bmatrix}.$$

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By Liouville's theorem, $f(\theta) - g(\theta)$ is constant, and since f and g are odd functions of θ, the constant is zero.

Theorem

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Then

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Let ω_3 and ω_4 be defined by $\begin{bmatrix}\omega_4\\ \omega_3\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\omega_2\\ \omega_1\end{bmatrix}$.

Then

$$Z(\theta, \omega_3, \omega_4) = Z(\theta, \omega_1, \omega_2) + \frac{2\pi i c\theta}{\omega_1\omega_3}.$$

Note: if $\tau = \dfrac{\omega_2}{\omega_1}$ then $\dfrac{\omega_4}{\omega_3} = \dfrac{a\tau + b}{c\tau + d}$ and $\text{Im}(\omega_4/\omega_3) > 0$.

Let E_k be the Eisenstein series defined by

$$E_k(\tau) = -\frac{B_k}{2k} + \sum_{j=1}^{\infty} \frac{j^{k-1} q^j}{1 - q^j}$$

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Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. (It means a, b, c, $d \in \mathbb{Z}$ and $ad - bc = 1$.)
Then

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + \frac{ic(c\tau + d)}{4\pi}$$

and

$$E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} E_{2k}(\tau), \qquad k = 2, 3, \ldots.$$

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We have
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Expand in powers of θ to get

$$\frac{1}{\theta} + \frac{4\pi}{\omega_3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}E_{2k}\left(\frac{\omega_4}{\omega_3}\right)\left(\frac{2\pi\theta}{\omega_3}\right)^{2k-1}$$

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Equate coefficients. Easy, when $k = 2, 3, 4, \ldots$.

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An additional term occurs for E_2 when $k = 1$.

Note: $\omega_3/\omega_1 = (c\omega_2 + d\omega_1)/\omega_1 = c\tau + d$.

Modular form of weight k: for all a, b, c, $d \in \mathbb{Z}$ with $ad - bc = 1$,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

plus some analyticity requirements (see a book on modular forms).

Examples:

$$q = \exp(2\pi i\tau) \qquad \text{and} \qquad r = \exp\left(2\pi i\ \frac{a\tau + b}{c\tau + d}\right).$$

Then

$$\begin{aligned}
P(r) &= (c\tau + d)^2 P(q) - \frac{6ic(c\tau + d)}{\pi} \quad \text{quasi-modular} \\
Q(r) &= (c\tau + d)^4 Q(q) \\
R(r) &= (c\tau + d)^6 R(q),
\end{aligned}$$

where

$$P(q) = 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1 - q^j}, \qquad Q(q) = 1 + 240\sum_{j=1}^{\infty} \frac{j^3 q^j}{1 - q^j}$$

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Hints: recall that $P(r) = (c\tau + d)^2 P(q) - \frac{6ic(c\tau + d)}{\pi}$

Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ so that

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Show that $\sum_{k=1}^{\infty} \frac{k}{e^{2\pi k}-1} = \frac{1}{24} - \frac{1}{8\pi}$. (Posed by Ramanujan)

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Then put $\tau = i$ and observe that $q = r = \exp(-2\pi)$.

Another example:

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From the previous lecture,

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It follows that $q \prod_{j=1}^{\infty} (1 - q^j)^{24}$ is a modular form of weight 12.

Dedekind's eta function is defined for $\mathrm{Im}\tau > 0$ by

$$\eta(\tau) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j) \quad \text{where} \quad q = \exp(2\pi i \tau).$$

For any integers a, b, c, d with $ad - bc = 1$, we have

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and so

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\, \eta(\tau)$$

(Why "i" in the square root term? Consider τ purely imaginary.)

Exercise: (the effect of scaling)
Suppose $f(\tau)$ is a modular form of weight k.
In particular, this means

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

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(End of lecture 2)