The inversion theorem for elliptic functions

Shaun Cooper

Massey University, Auckland, New Zealand
Lecture 3: Inversion for elliptic functions.
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Let’s begin with a 3-page motivation.
Recall Theorem 2:

\[ s(\theta - \alpha)s(\theta + \alpha)s(\beta - \gamma)s(\beta + \gamma) \]
\[ + s(\theta - \beta)s(\theta + \beta)s(\gamma - \alpha)s(\gamma + \alpha) \]
\[ + s(\theta - \gamma)s(\theta + \gamma)s(\alpha - \beta)s(\alpha + \beta) = 0. \]
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+ s(\theta - \beta)s(\theta + \beta)s(\gamma - \alpha)s(\gamma + \alpha) \\
+ s(\theta - \gamma)s(\theta + \gamma)s(\alpha - \beta)s(\alpha + \beta) = 0.
\]

This can be written in explicit form.

Let \( a^2 = bcde \) and suppose \(|q| < 1\). Then

\[
[a/b; q]_\infty [a/c; q]_\infty [a/d; q]_\infty [a/e; q]_\infty - [b; q]_\infty [c; q]_\infty [d; q]_\infty [e; q]_\infty \\
= b [a; q]_\infty [a/bc; q]_\infty [a/bd; q]_\infty [a/be; q]_\infty
\]

where

\[
[x; q]_\infty = \prod_{j=1}^{\infty} (1 - q^{j-1}x)(1 - q^jx)
\]

is the product with zeros on a bilateral geometric progression.
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\[
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\]

\[
= b \left[a; q\right]_{\infty} \left[a/bc; q\right]_{\infty} \left[a/bd; q\right]_{\infty} \left[a/be; q\right]_{\infty}
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where

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Replace $q$ with $q^2$, then put $a = -q^2$, $b = c = d = e = -q$ to get
Let $a^2 = bcde$ and suppose $|q| < 1$. Then

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Replace $q$ with $q^2$, then put $a = -q^2$, $b = c = d = e = -q$ to get

$$\prod_{j=1}^{\infty} (1 + q^{2j-1})^8 - \prod_{j=1}^{\infty} (1 - q^{2j-1})^8 = 16q \prod_{j=1}^{\infty} (1 + q^{2j})^8$$

Jacobi called this “aequatio identica satis abstrusa” (nonobvious identity).
Rewrite Jacobi’s identity in the form

\[ 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^8}{(1 + q^{2j-1})^8} + \prod_{j=1}^{\infty} \frac{(1 - q^{2j-1})^8}{(1 + q^{2j-1})^8} = 1 \]
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We will prove: \( x \) increases from 0 to 1 as \( q \) increases from 0 to 1.
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\[
q = \exp \left( -\pi \frac{F(1 - x)}{F(x)} \right), \quad \text{where} \quad F(x) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left( \frac{x}{16} \right)^n.
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Moreover,

\[ \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left( \frac{x}{16} \right)^n = \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^2. \]
Let \( x = 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^8}{(1 + q^{2j-1})^8}. \)

Observe, that \( x = 0 \) when \( q = 0 \).
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By Jacobi’s identity, \( 1 - x = \prod_{j=1}^{\infty} \frac{(1 - q^{2j-1})^8}{(1 + q^{2j-1})^8} \).
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This shows that \( 1 - x \) is a decreasing function for \( 0 < q < 1 \).
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This shows that $1 - x$ is a decreasing function for $0 < q < 1$.
And, as $q \to 1^-$ it is clear $1 - x \to 0$. 
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Hence, \( x \) increases from 0 to 1 as \( q \) increases from 0 to 1.

This is the first of several properties we will need to know about \( x. \)
\[ x = 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^8}{(1 + q^{2j-1})^8} \]
\[
\begin{align*}
x &= 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^8}{(1 + q^{2j-1})^8} \times \frac{(1 + q^{2j})^8}{(1 + q^{2j})^8} \\
&= 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^8}{(1 + q^{2j})^8} \times \frac{(1 + q^{2j})^8}{(1 + q^{2j})^8}
\end{align*}
\]
x = 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^8}{(1 + q^{2j-1})^8} \times \frac{(1 + q^{2j})^8}{(1 + q^{2j})^8} \\
= 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^{16}}{(1 + q^j)^8}
\[
\times = 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^8}{(1 + q^{2j-1})^8} \times \frac{(1 + q^{2j})^8}{(1 + q^{2j})^8}
\]

\[
= 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^{16}}{(1 + q^{j})^8}
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\]
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\[ = 16q \prod_{j=1}^{\infty} \frac{(1 - q^{4j})^{16}}{(1 - q^{2j})^8} \times \frac{(1 - q^{j})^8}{(1 - q^{2j})^{16}} \]
\[
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\]

\[
= 16q \prod_{j=1}^{\infty} \frac{(1 - q^{4j})^{16}(1 - q^{j})^8}{(1 - q^{2j})^{24}}
\]
\[
\times = 16q \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^8}{(1 + q^{2j-1})^8} \times \frac{(1 + q^{2j})^8}{(1 + q^{2j})^8} \\
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= 16q \prod_{j=1}^{\infty} \frac{(1 - q^{4j})^{16}(1 - q^j)^8}{(1 - q^{2j})^{24}} \\
= 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)}
\]
From the previous page, we have

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In a similar way, it may be shown that

\[ 1 - x = \frac{\eta^{16}(\tau)\eta^8(4\tau)}{\eta^{24}(2\tau)} \]
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$$x = 16 \frac{\eta^{16}(4\tau)\eta^{8}(\tau)}{\eta^{24}(2\tau)}$$

In a similar way, it may be shown that

$$1 - x = \frac{\eta^{16}(\tau)\eta^{8}(4\tau)}{\eta^{24}(2\tau)}$$

and

$$z := \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^2 = \frac{\eta^{10}(2\tau)}{\eta^{4}(\tau)\eta^{4}(4\tau)} \quad \text{(use Jacobi triple product)}$$

where (recall that) \( \eta(\tau) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j) \)

and the symbol “:=” denotes a definition.
Canonical notation:

\[ x = 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)}, \quad 1-x = \frac{\eta^{16}(\tau)\eta^8(4\tau)}{\eta^{24}(2\tau)}, \quad z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)}. \]
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Note the symmetries.
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Note the symmetries.

Weights: \( x \) and \( 1-x \) have weight 0, whereas \( z \) has weight 1.
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**Exercise:** Use the transformation formula

\[ \eta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \eta(\tau) \]

to prove that for any real positive number \( t \),

\[ x(e^{-\pi t}) + x(e^{-\pi/t}) = 1 \]

and deduce that \( x(e^{-\pi}) = \frac{1}{2}. \)

Hint: Let \( \tau = it/2. \)
\begin{align*}
x &= 16 \frac{\eta^{16}(4\tau)\eta^{8}(\tau)}{\eta^{24}(2\tau)}, \\
1-x &= \frac{\eta^{16}(\tau)\eta^{8}(4\tau)}{\eta^{24}(2\tau)}, \\
z &= \frac{\eta^{10}(2\tau)}{\eta^{4}(\tau)\eta^{4}(4\tau)}.
\end{align*}

Derivatives:
\begin{align*}
x &= 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)}, \quad 1-x = \frac{\eta^{16}(\tau)\eta^8(4\tau)}{\eta^{24}(2\tau)}, \quad z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)}.
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Derivatives:

\begin{align*}
q \frac{d}{dq} \log \left( \frac{x}{1-x} \right) &= q \frac{d}{dq} \log \left( 16 \frac{\eta^8(4\tau)}{\eta^8(\tau)} \right)
\end{align*}
\[ x = 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)} , \quad 1-x = \frac{\eta^{16}(\tau)\eta^8(4\tau)}{\eta^{24}(2\tau)} , \quad z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)} . \]

Derivatives:

\[ q \frac{d}{dq} \log \left( \frac{x}{1-x} \right) = q \frac{d}{dq} \log \left( 16 \frac{\eta^8(4\tau)}{\eta^8(\tau)} \right) = \frac{1}{3} \left( 4P(q^4) - P(q) \right) . \]
\[ x = 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)}, \quad 1-x = \frac{\eta^{16}(\tau)\eta^8(4\tau)}{\eta^{24}(2\tau)}, \quad z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)}. \]

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\[ = \frac{1}{3} (4P(q^4) - P(q)). \]

Note:

\[ q \frac{d}{dq} \log \eta^{24}(\tau) = q \frac{d}{dq} \log \left( q \prod_{j=1}^{\infty} (1 - q^j)^{24} \right) \]
\[ x = 16 \frac{\eta^{16}(4\tau) \eta^8(\tau)}{\eta^{24}(2\tau)}, \quad 1-x = \frac{\eta^{16}(\tau) \eta^8(4\tau)}{\eta^{24}(2\tau)}, \quad z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau) \eta^4(4\tau)}. \]

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Note:

\[
q \frac{d}{dq} \log \eta^{24}(\tau) = q \frac{d}{dq} \log \left( q \prod_{j=1}^{\infty} (1-q^j)^{24} \right)
= 1 - 24 \sum_{j=1}^{\infty} \frac{j q^j}{1-q^j} = P(q).
\]
Canonical notation:

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\[ = \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^4 \] (by sum of four squares)
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\[ = z^2 \quad \text{(by definition)}. \]
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It follows that

\[ q \frac{d}{dq} x = z^2 x (1-x). \]
Parameterizations in terms of $z$ and $x$

\[ x = 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)}, \quad 1-x = \frac{\eta^{16}(\tau)\eta^8(4\tau)}{\eta^{24}(2\tau)}, \quad z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)}. \]
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By rearrangement,

\[ 16\eta^{24}(\tau) = z^{12}x(1-x)^4, \]
\[ 16^2\eta^{24}(2\tau) = z^{12}x^2(1-x)^2, \]
\[ 16^4\eta^{24}(4\tau) = z^{12}x^4(1-x). \]
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Take logarithms and differentiate, to obtain (for example)

$$P(q) = \frac{12}{z} q \frac{dz}{dq} + (1 - 5x)z^4.$$
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$$P(q) = \frac{12}{z} q \frac{dz}{dq} + (1-5x)z^4.$$ 

Differentiate again (what do we need to know?)
\[ x = 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)}, \quad z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)}. \]

We eventually obtain

\[ \frac{d}{dx} \left( x(1-x) \frac{dz}{dx} \right) = \frac{z}{4}. \]
$$x = 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)}, \quad z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)}.$$ 

We eventually obtain

$$\frac{d}{dx} \left( x(1 - x) \frac{dz}{dx} \right) = \frac{z}{4}.$$ 

**Exercise:** Let $z_1 = z \log q$. Show that

$$\frac{d}{dx} \left( x(1 - x) \frac{dz_1}{dx} \right) = \frac{z_1}{4}.$$
Given: \( x = 16 \frac{\eta^{16}(4\tau)\eta^{8}(\tau)}{\eta^{24}(2\tau)} \) and \( z = \frac{\eta^{10}(2\tau)}{\eta^{4}(\tau)\eta^{4}(4\tau)} \),

we have \( \frac{d}{dx} \left(x(1 - x)\frac{dz}{dx}\right) = \frac{z}{4} \).
Given: \( x = 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)} \) and \( z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)} \),

we have \( \frac{d}{dx} \left( x(1-x) \frac{dz}{dx} \right) = \frac{z}{4} \).

The general solution is

\[
z = A_2 F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right) + B_2 F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1-x \right).
\]
Given: \[ x = 16 \frac{\eta^{16}(4\tau)\eta^8(\tau)}{\eta^{24}(2\tau)} \] and \[ z = \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)}, \]

we have \[ \frac{d}{dx} \left( x(1 - x) \frac{dz}{dx} \right) = \frac{z}{4}. \]

The general solution is

\[ z = A_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right) + B_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - x \right). \]

When \( q = 0 \), we have \( x = 0 \) and \( z = 1 \).
Given: \( x = 16 \frac{\eta^{16}(4\tau)\eta^{8}(\tau)}{\eta^{24}(2\tau)} \) and \( z = \frac{\eta^{10}(2\tau)}{\eta^{4}(\tau)\eta^{4}(4\tau)} \),

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Given: \[ x = 16 \frac{\eta^{16}(4\tau)}{\eta^{24}(2\tau)} \text{ and } z = \frac{\eta^{10}(2\tau)}{\eta^{4}(\tau)\eta^{4}(4\tau)}, \]

we have \[ \frac{d}{dx} \left( x(1 - x) \frac{dz}{dx} \right) = \frac{z}{4}. \]

The general solution is

\[ z = A_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) + B_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x\right). \]

When \( q = 0 \), we have \( x = 0 \) and \( z = 1 \).

It follows that \( A = 1 \) and \( B = 0 \).

We conclude that

\[ z = 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right). \]
We also have \( \frac{d}{dx} \left( x(1 - x) \frac{dz_1}{dx} \right) = \frac{z_1}{4} \) where \( z_1 = z \log q \).
We also have \[ \frac{d}{dx} \left( x(1-x) \frac{dz_1}{dx} \right) = \frac{z_1}{4} \] where \( z_1 = z \log q \).

The general solution is

\[ z_1 = z \log q = C \, _2F_1\left( \frac{1}{2}, \frac{1}{2}; 1; x \right) + D \, _2F_1\left( \frac{1}{2}, \frac{1}{2}; 1; 1 - x \right). \]
We also have \[
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\]

Divide by \(z\), to deduce
\[
\log q = C + D \frac{\, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x\right)}{\, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}.
\]
We also have \[
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Divide by \(z\), to deduce

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\]

When \(q \to 1^-\) we have \(x \to 1\) and \(z \to \infty\). It follows that \(C = 0\).
We also have \[ \frac{d}{dx} \left( x(1 - x) \frac{dz_1}{dx} \right) = \frac{z_1}{4} \] where \( z_1 = z \log q \).

The general solution is

\[ z_1 = z \log q = C_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) + D_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x\right). \]

Divide by \( z \), to deduce

\[ \log q = C + D \frac{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}. \]

When \( q \rightarrow 1^- \) we have \( x \rightarrow 1 \) and \( z \rightarrow \infty \). It follows that \( C = 0 \).

When \( q = e^{-\pi} \), we have \( x = \frac{1}{2} \) and it follows that \( D = -\pi \).
We also have  \( \frac{d}{dx} \left( x(1-x) \frac{dz_1}{dx} \right) = z_1/4 \) where \( z_1 = z \log q \).

The general solution is

\[
z_1 = z \log q = C \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) + D \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right).
\]

Divide by \( z \), to deduce

\[
\log q = C + D \frac{2 \, _1F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{2 \, _1F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}.
\]

When \( q \to 1^- \) we have \( x \to 1 \) and \( z \to \infty \). It follows that \( C = 0 \).

When \( q = e^{-\pi} \), we have \( x = \frac{1}{2} \) and it follows that \( D = -\pi \).

We conclude that \( q = \exp \left( -\pi \frac{2 \, _1F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{2 \, _1F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)} \right) \).
A basic example of parameterization:

The circle $x^2 + y^2 = 1$ can be parameterized by the trigonometric functions

$$x = \cos t \quad \text{and} \quad y = \sin t.$$
Summary:

The hypergeometric function \( z = _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \) can be parameterized by modular forms:
Summary:

The hypergeometric function $z = \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$ can be parameterized by modular forms:

$$z = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2$$

and

$$x = \left(\frac{\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}}\right)^4.$$
Summary:

The hypergeometric function $z = 2F_1\left( \frac{1}{2}, \frac{1}{2}; 1; x \right)$ can be parameterized by modular forms:

$$z = \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^2 \quad \text{and} \quad x = \left( \frac{\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \right)^4 .$$

Moreover, the inverse function exists and is given by

$$q = \exp \left( -\pi \frac{2F_1\left( \frac{1}{2}, \frac{1}{2}; 1; 1 - x \right)}{2F_1\left( \frac{1}{2}, \frac{1}{2}; 1; x \right)} \right) .$$
Ramanujan: Modular Equations and Approximations to $\pi$

14. The ordinary modular equations express the relations which hold between $k$ and $l$ when $nK'/K = L'/L$, or $q^n = Q$, where

$$q = e^{-\pi K/K}, \quad Q = e^{-\pi L/L},$$

$$K = 1 + \left(\frac{1}{2}\right)k^2 + \left(\frac{1}{2, 4}\right)k^4 + \ldots$$

There are corresponding theories in which $q$ is replaced by one or other of the functions

$$q_1 = e^{-\pi K_1/\sqrt{K_1}}, \quad q_2 = e^{-2\pi K_2/(K_1\sqrt{3})}, \quad q_3 = e^{-2\pi K_3/\sqrt{K_3}},$$

where

$$K_1 = 1 + \frac{1.3}{4^2}k^2 + \frac{1.3 \cdot 5.7}{4^4 \cdot 8^2}k^4 + \frac{1.3 \cdot 5.7 \cdot 9.11}{4^6 \cdot 8^4 \cdot 12^2}k^6 + \ldots,$$

$$K_2 = 1 + \frac{1.2}{3^2}k^2 + \frac{1.2 \cdot 4.5}{3^4 \cdot 6^2}k^4 + \frac{1.2 \cdot 4.5 \cdot 7.8}{3^6 \cdot 6^4 \cdot 9^2}k^6 + \ldots,$$

$$K_3 = 1 + \frac{1.5}{6^2}k^2 + \frac{1.5 \cdot 7.11}{6^4 \cdot 12^2}k^4 + \frac{1.5 \cdot 7.11 \cdot 13.17}{6^6 \cdot 12^4 \cdot 18^2}k^6 + \ldots$$

From these theories we can deduce further series for $1/\pi$, such as

$$\frac{27}{4\pi} = 2 + 17 \frac{1}{23} + \frac{1}{3} \frac{2}{27}$$

$$+ 32 \frac{1.3}{2.4} \frac{1.4}{5} \frac{1}{27} \frac{2}{27} + \ldots \ldots \ldots (31)$$

$$\frac{15 \sqrt{3}}{2\pi} = 4 + 37 \frac{1}{23} + \frac{1}{3} \frac{4}{125}$$

$$+ 70 \frac{1.3}{2.4} \frac{1.4}{5} \frac{1}{125} \frac{4}{125} + \ldots \ldots \ldots (32)$$

$$\frac{5 \sqrt{5}}{2\pi \sqrt{3}} = 1 + 12 \frac{1}{26} + \frac{1}{6} \frac{4}{125}$$

$$+ 23 \frac{1.3}{2.4} \frac{1.7}{5} \frac{11}{6} \frac{4}{125} + \ldots \ldots \ldots (33)$$

$$\frac{85 \sqrt{85}}{18\pi \sqrt{3}} = 8 + 141 \frac{1}{26} + \frac{1}{6} \frac{4}{85}$$

$$+ 274 \frac{1.3}{2.4} \frac{1.7}{5} \frac{11}{6} \frac{4}{85} + \ldots \ldots \ldots (34)$$
There are similar theories when

\[ q = \exp \left( -\pi \frac{z(1-x)}{z(x)} \right), \quad z(x) = \, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right) \]

is replaced by any of

\[ q_1 = \exp \left( -\pi \sqrt{2} \frac{z_1(1-x)}{z_1(x)} \right), \quad z_1(x) = \, _2F_1 \left( \frac{1}{4}, \frac{3}{4}; 1; x \right) \]

\[ q_2 = \exp \left( -\frac{2\pi}{\sqrt{3}} \frac{z_2(1-x)}{z_2(x)} \right), \quad z_2(x) = \, _2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; x \right) \]

\[ q_3 = \exp \left( -2\pi \frac{z_3(1-x)}{z_3(x)} \right), \quad z_3(x) = \, _2F_1 \left( \frac{1}{6}, \frac{5}{6}; 1; x \right) \]
Ramanujan (1914):

\[
q = \exp \left( -\frac{2\pi}{\sqrt{3}} \frac{2\, F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; 1 - x \right)}{2\, F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; x \right)} \right)
\]
Ramanujan (1914):

\[ q = \exp \left( -\frac{2\pi}{\sqrt{3}} \frac{2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; 1 - x \right)}{2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; x \right)} \right) \]


\[ x = \left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2 + (m+\frac{1}{3})(n+\frac{1}{3}) + (n+\frac{1}{3})^2} \right)^3 \]

\[ x = \left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2 + mn + n^2} \right)^3 \]
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Ramanujan’s “alternative theories” of elliptic functions

Ramanujan

- pp. 257–262, second notebook
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- 1914 paper “Modular equations and approximations to $1/\pi$”
  17 series for $1/\pi$

Mordell (1927), Watson (1931)

- “It is unfortunate that Ramanujan has not developed in detail the corresponding theories...”
- “There are developments of functions analogous to elliptic functions which I have not seen elsewhere...”
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Initial investigations into the “alternative theories”
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Ramanujan’s “alternative theories” of elliptic functions


The $\text{2F}_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)$ theory
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Ramanujan’s “alternative theories” of elliptic functions

The $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)$ theory

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The $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)$ theory

C., (2009)
A unified treatment for all four theories
By Clausen’s identity,

\[ z^2 = 2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right)^2 = 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; 4x(1-x) \right) \]

\[ = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left( \frac{x(1-x)}{16} \right)^j. \]
By Clausen’s identity,

\[ z^2 = 2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right)^2 = 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; 4x(1 - x) \right) = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left( \frac{x(1 - x)}{16} \right)^j. \]

Write \( Z = z^2 \) and \( X = x(1 - x)/16 \) so that

\[ Z = \sum_{j=0}^{\infty} \binom{2j}{j}^3 X^j. \]
The series

\[ Z = \sum_{j=0}^{\infty} \binom{2j}{j}^3 X^j. \]

is parameterized by

\[ Z = \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^4 = \frac{1}{3} \left( 4(P(q^4) - P(q)) \right) \]

and

\[ X = \frac{\eta_1^4 \eta_4^4}{\eta_2^4 Z} = \frac{\eta_1^{24} \eta_4^{24}}{\eta_2^{48}} \]
The series

\[ Z = \sum_{j=0}^{\infty} \binom{2j}{j}^3 X^j. \]

is parameterized by

\[ Z = \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^4 = \frac{1}{3} \left( 4(P(q^4) - P(q)) \right) \]

and

\[ X = \frac{\eta_4^4 \eta_4^4}{\eta_2^4 Z} = \frac{\eta_1^{24} \eta_4^{24}}{\eta_2^{48}} \]

where

\[ P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}, \quad \eta_k = q^{k/24} \prod_{n=1}^{\infty} (1 - q^{nk}). \]
\[ f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left( \frac{\eta_1^4 \eta_4^4}{\eta_2^4 f(4)} \right)^{2j} \]
\[
f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left( \frac{\eta_1^4 \eta_4^4}{\eta_2^4 f(4)} \right)^{2j}
\]

\[
f(3) := \frac{3P(q^3) - P(q)}{2} = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{3j}{j} \left( \frac{\eta_1^2 \eta_3^2}{f(3)} \right)^{3j}
\]
\[ f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left( \frac{\eta_1^4 \eta_4^4}{\eta_2^4 f(4)} \right)^{2j} \]

\[ f(3) := \frac{3P(q^3) - P(q)}{2} = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{3j}{j} \left( \frac{\eta_1^2 \eta_3^2}{f(3)} \right)^{3j} \]

\[ f(2) := 2P(q^2) - P(q) = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{4j}{2j} \left( \frac{\eta_1^2 \eta_2^2}{f(2)} \right)^{4j} \]
\[ f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left( \frac{\eta_1^4 \eta_4^4}{\eta_2^4 f(4)} \right)^{2j} \]

\[ f(3) := \frac{3P(q^3) - P(q)}{2} = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{3j}{j} \left( \frac{\eta_1^2 \eta_3^2}{f(3)} \right)^{3j} \]

\[ f(2) := 2P(q^2) - P(q) = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{4j}{2j} \left( \frac{\eta_1^2 \eta_2^2}{f(2)} \right)^{4j} \]

\[ f(1) := Q(q)^{1/2} = \sum_{j=0}^{\infty} \binom{2j}{j} \binom{3j}{j} \binom{6j}{3j} \left( \frac{\eta_1^4}{f(1)} \right)^{6j} \]

\[ P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1 - q^j}, \quad Q(q) = 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1 - q^j} \]

\[ \eta_m = q^{m/24} \prod_{j=1}^{\infty} (1 - q^{mj}) \]
Higher levels

\[ f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left( \frac{\eta_1^4 \eta_4^4}{\eta_2^4 f(4)} \right)^{2j} \]

\[ f(5) := \frac{5P(q^5) - P(q)}{4} = \sum_{j=0}^{\infty} \binom{2j}{j} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j + k}{k} \right\} \left( \frac{\eta_1^2 \eta_5^2}{f(5)} \right)^{2j} \]
Higher levels

\[ f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left( \frac{\eta_1^4 \eta_4^4}{\eta_2^4 f(4)} \right)^{2j} \]

\[ f(5) := \frac{5P(q^5) - P(q)}{4} = \sum_{j=0}^{\infty} \binom{2j}{j} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k} \right\} \left( \frac{\eta_1^2 \eta_5^2}{f(5)} \right)^{2j} \]

\[ s_j = \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k} \]

\[ (j + 1)^2 s_{j+1} = (11j^2 + 11j + 3)s_j + j^2 s_{j-1} \]

R. Apéry: \( \zeta(2) \notin \mathbb{Q} \)
Rogers-Ramanujan continued fraction

\[
\frac{5P(q^5) - P(q)}{4} = \sum_{j=0}^{\infty} \binom{2j}{j} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k} \right\} \left( \frac{\eta_1^2 \eta_5^2}{f(5)} \right)^{2j}
\]
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\frac{5P(q^5) - P(q)}{4} = \sum_{j=0}^{\infty} \binom{2j}{j} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k} \right\} \left( \frac{\eta_1^2 \eta_5^2}{f(5)} \right)^{2j}
\]

\[
r = r(q) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}}.
\]

\[
\left( \frac{\eta_1^2 \eta_5^2}{f(5)} \right)^2 = \frac{r^5(1 - 11r^5 - r^{10})}{(1 + r^{10})^2}.
\]

The Rogers-Ramanujan continued fraction is an analogue of the elliptic modulus \( k \).
More precisely, \( r^5/(1 + r^{10}) \) plays the role of \( k^2 \).
Summary of Lecture 3:

The hypergeometric function
\[ z = \binom{1}{2} \binom{1}{2} \binom{1}{x} \]
can be parameterized by the modular forms
\[ z = \left( \sum_{n=-\infty}^{\infty} q^n \right)^2 \]
and
\[ x = \left( \sum_{n=-\infty}^{\infty} q^n \left( n + \frac{1}{2} \right)^2 \right) \left( \sum_{n=-\infty}^{\infty} q^n \right)^2. \]

Inverse:
\[ q = \exp \left( -\frac{\pi^2}{2} \binom{1}{2} \binom{1}{2} \binom{1}{1-x} \right) \binom{1}{2} \binom{1}{2} \binom{1}{x} \]

This is the level 4 theory, developed by Jacobi, before "levels" were invented.

Similar theories: levels \( 1 \leq \ell \leq 18, 20, 21, 22, 23, 24, 25, 33, 35. \)

E.g., Cooper's book has levels \( 1 \leq \ell \leq 12. \)

Other levels are open to investigation. (End of lecture 3)
Summary of Lecture 3:

The hypergeometric function \( z = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \) can be parameterized by the modular forms

\[
z = \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^2 \quad \text{and} \quad x = \left( \frac{\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \right)^4.
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Summary of Lecture 3:

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Inverse: \( q = \exp \left( -\pi \frac{\, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x\right)}{\, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)} \right) \).
Summary of Lecture 3:

The hypergeometric function $z = 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$ can be parameterized by the modular forms

$$z = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 \quad \text{and} \quad x = \left(\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}\right)^4$$

Inverse: $q = \exp\left(-\pi \frac{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}\right)$.

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