The inversion theorem for elliptic functions

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Lecture 3: Inversion for elliptic functions.

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Let's begin with a 3-page motivation.

Recall Theorem 2:

$$\begin{aligned} & s(\theta-\alpha)s(\theta+\alpha)s(\beta-\gamma)s(\beta+\gamma) \\ & + s(\theta-\beta)s(\theta+\beta)s(\gamma-\alpha)s(\gamma+\alpha) \\ & + s(\theta-\gamma)s(\theta+\gamma)s(\alpha-\beta)s(\alpha+\beta) \quad = \quad 0. \end{aligned}$$

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This can be written in explicit form.

Let $a^2 = bcde$ and suppose $|q| < 1$. Then

$$[a/b;q]_\infty\,[a/c;q]_\infty\,[a/d;q]_\infty\,[a/e;q]_\infty - [b;q]_\infty\,[c;q]_\infty\,[d;q]_\infty\,[e;q]_\infty$$

$$= b\,[a;q]_\infty\,[a/bc;q]_\infty\,[a/bd;q]_\infty\,[a/be;q]_\infty$$

where

$$[x;q]_\infty = \prod_{j=1}^{\infty}(1-q^{j-1}x)(1-q^j x)$$

is the product with zeros on a bilateral geometric progression.

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$$\prod_{j=1}^{\infty}(1+q^{2j-1})^8 - \prod_{j=1}^{\infty}(1-q^{2j-1})^8 = 16q\prod_{j=1}^{\infty}(1+q^{2j})^8$$

Jacobi called this "aequatio identica satis abstrusa" (nonobvious identity).

Rewrite Jacobi's identity in the form

$$16q\prod_{j=1}^{\infty}\frac{(1+q^{2j})^8}{(1+q^{2j-1})^8}+\prod_{j=1}^{\infty}\frac{(1-q^{2j-1})^8}{(1+q^{2j-1})^8}=1$$

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$$q=\exp\left(-\pi\,\frac{F(1-x)}{F(x)}\right),\quad \text{where}\quad F(x)=\sum_{n=0}^{\infty}\binom{2n}{n}^2\left(\frac{x}{16}\right)^n.$$

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Moreover,

$$\sum_{n=0}^{\infty}\binom{2n}{n}^2\left(\frac{x}{16}\right)^n=\left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)^2.$$

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This is the first of several properties we will need to know about x.

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&= 16q \prod_{j=1}^{\infty} \frac{(1-q^{4j})^{16}}{(1-q^{2j})^8} \times \frac{(1-q^{j})^8}{(1-q^{2j})^{16}}
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&= 16q\prod_{j=1}^{\infty}\frac{(1-q^{4j})^{16}(1-q^{j})^8}{(1-q^{2j})^{24}}\\
&= 16\frac{\eta^{16}(4\tau)\eta^{8}(\tau)}{\eta^{24}(2\tau)}
\end{aligned}$$

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and

$$z := \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 = \frac{\eta^{10}(2\tau)}{\eta^{4}(\tau)\eta^{4}(4\tau)} \quad \text{(use Jacobi triple product)}$$

where (recall that) $\quad \eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1 - q^j)$

and the symbol ":=" denotes a definition.

Canonical notation:

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Note the symmetries.

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Weights: x and $1-x$ have weight 0, whereas z has weight 1.

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Exercise: Use the transformation formula

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau)$$

to prove that for any real positive number t,

$$x(e^{-\pi t}) + x(e^{-\pi/t}) = 1$$

and deduce that $x(e^{-\pi}) = \frac{1}{2}$.

Hint: Let $\tau = it/2$.

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Derivatives:

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Derivatives:

$$q\frac{\mathrm{d}}{\mathrm{d}q}\log\left(\frac{x}{1-x}\right) = q\frac{\mathrm{d}}{\mathrm{d}q}\log\left(16\frac{\eta^{8}(4\tau)}{\eta^{8}(\tau)}\right)$$

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Note: $$q\frac{\mathrm{d}}{\mathrm{d}q}\log \eta^{24}(\tau) = q\frac{\mathrm{d}}{\mathrm{d}q}\log\left(q\prod_{j=1}^{\infty}(1-q^j)^{24}\right)$$

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It follows that

$$q\frac{\mathrm{d}x}{\mathrm{d}q} = z^2 x(1-x).$$

Parameterizations in terms of z and x

$$x = 16\frac{\eta^{16}(4\tau)\eta^{8}(\tau)}{\eta^{24}(2\tau)}, \qquad 1-x = \frac{\eta^{16}(\tau)\eta^{8}(4\tau)}{\eta^{24}(2\tau)}, \quad z = \frac{\eta^{10}(2\tau)}{\eta^{4}(\tau)\eta^{4}(4\tau)}.$$

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By rearrangement,

$$\begin{aligned} 16\eta^{24}(\tau) &= z^{12}x(1-x)^4 \\ 16^2\eta^{24}(2\tau) &= z^{12}x^2(1-x)^2 \\ 16^4\eta^{24}(4\tau) &= z^{12}x^4(1-x). \end{aligned}$$

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Take logarithms and differentiate, to obtain (for example)

$$P(q) = \frac{12}{z}\,q\frac{\mathrm{d}z}{\mathrm{d}q} + (1-5x)z^4.$$

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Differentiate again (what do we need to know?)

$$x = 16\frac{\eta^{16}(4\tau)\eta^{8}(\tau)}{\eta^{24}(2\tau)}, \quad z = \frac{\eta^{10}(2\tau)}{\eta^{4}(\tau)\eta^{4}(4\tau)}.$$

We eventually obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)\frac{\mathrm{d}z}{\mathrm{d}x}\right) = \frac{z}{4}.$$

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Exercise: Let $z_1 = z \log q$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)\frac{\mathrm{d}z_1}{\mathrm{d}x}\right) = \frac{z_1}{4}.$$

Given: $x = 16\frac{\eta^{16}(4\tau)\eta^{8}(\tau)}{\eta^{24}(2\tau)}$ and $z = \frac{\eta^{10}(2\tau)}{\eta^{4}(\tau)\eta^{4}(4\tau)}$,

we have $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)\frac{\mathrm{d}z}{\mathrm{d}x}\right) = \frac{z}{4}$.

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The general solution is

$$z = A\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;x) + B\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x).$$

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When $q = 0$, we have $x = 0$ and $z = 1$.

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It follows that $A = 1$ and $B = 0$.

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When $q = 0$, we have $x = 0$ and $z = 1$.

It follows that $A = 1$ and $B = 0$.

We conclude that

$$z = {}_2F_1(\frac{1}{2},\frac{1}{2};1;x).$$

We also have $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)\dfrac{\mathrm{d}z_1}{\mathrm{d}x}\right) = \dfrac{z_1}{4}$ where $z_1 = z \log q$.

We also have $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)\dfrac{\mathrm{d}z_1}{\mathrm{d}x}\right) = \dfrac{z_1}{4}$ where $z_1 = z \log q$.

The general solution is

$$z_1 = z \log q = C\,{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x) + D\,{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x).$$

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Divide by z, to deduce

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When $q \to 1^-$ we have $x \to 1$ and $z \to \infty$. It follows that $C = 0$.

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When $q \to 1^-$ we have $x \to 1$ and $z \to \infty$. It follows that $C = 0$.

When $q = e^{-\pi}$, we have $x = \frac{1}{2}$ and it follows that $D = -\pi$.

We conclude that $q = \exp\left(-\pi\,\dfrac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right)$.

A basic example of parameterization:

The circle $x^2 + y^2 = 1$ can be parameterized by the trigonometric functions

$$x = \cos t \quad \text{and} \quad y = \sin t.$$

Summary:

The hypergeometric function $z = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ can be parameterized by modular forms:

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Summary:

The hypergeometric function $z = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ can be parameterized by modular forms:

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Moreover, the inverse function exists and is given by

$$q = \exp\left(-\pi \, \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} \right).$$

14. The ordinary modular equations express the relations which hold between k and l when $nK'/K = L'/L$, or $q^n = Q$, where

$$q = e^{-\pi K'/K}, \quad Q = e^{-\pi L'/L},$$

$$K = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 k^4 + \ldots.$$

There are corresponding theories in which q is replaced by one or other of the functions

$$q_1 = e^{-\pi K_1'\sqrt{2}/K_1}, \quad q_2 = e^{-2\pi K_2'/(K_2\sqrt{3})}, \quad q_3 = e^{-2\pi K_3'/K_3},$$

where

$$K_1 = 1 + \frac{1\,.\,3}{4^2}k^2 + \frac{1\,.\,3\,.\,5\,.\,7}{4^2\,.\,8^2}k^4 + \frac{1\,.\,3\,.\,5\,.\,7\,.\,9\,.\,11}{4^2\,.\,8^2\,.\,12^2}k^6 + \ldots,$$

$$K_2 = 1 + \frac{1\,.\,2}{3^2}k^2 + \frac{1\,.\,2\,.\,4\,.\,5}{3^2\,.\,6^2}k^4 + \frac{1\,.\,2\,.\,4\,.\,5\,.\,7\,.\,8}{3^2\,.\,6^2\,.\,9^2}k^6 + \ldots,$$

$$K_3 = 1 + \frac{1\,.\,5}{6^2}k^2 + \frac{1\,.\,5\,.\,7\,.\,11}{6^2\,.\,12^2}k^4 + \frac{1\,.\,5\,.\,7\,.\,11\,.\,13\,.\,17}{6^2\,.\,12^2\,.\,18^2}k^6 + \ldots.$$

From these theories we can deduce further series for $1/\pi$, such as

$$\frac{27}{4\pi} = 2 + 17\,\frac{1}{2}\,\frac{1}{3}\,\frac{2}{3}\left(\frac{2}{27}\right) + 32\,\frac{1\,.\,3}{2\,.\,4}\,\frac{1\,.\,4}{3\,.\,6}\,\frac{2\,.\,5}{3\,.\,6}\left(\frac{2}{27}\right)^2 + \ldots, \quad \ldots\ldots(31)$$

$$\frac{15\sqrt{3}}{2\pi} = 4 + 37\,\frac{1}{2}\,\frac{1}{3}\,\frac{2}{3}\left(\frac{4}{125}\right) + 70\,\frac{1\,.\,3}{2\,.\,4}\,\frac{1\,.\,4}{3\,.\,6}\,\frac{2\,.\,5}{3\,.\,6}\left(\frac{4}{125}\right)^2 + \ldots, \quad \ldots\ldots(32)$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}} = 1 + 12\,\frac{1}{2}\,\frac{1}{6}\,\frac{5}{6}\left(\frac{4}{125}\right) + 23\,\frac{1\,.\,3}{2\,.\,4}\,\frac{1\,.\,7}{6\,.\,12}\,\frac{5\,.\,11}{6\,.\,12}\left(\frac{4}{125}\right)^2 + \ldots, \quad \ldots(33)$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = 8 + 141\,\frac{1}{2}\,\frac{1}{6}\,\frac{5}{6}\left(\frac{4}{85}\right)^3 + 274\,\frac{1\,.\,3}{2\,.\,4}\,\frac{1\,.\,7}{6\,.\,12}\,\frac{5\,.\,11}{6\,.\,12}\left(\frac{4}{85}\right)^6 + \ldots, \quad \ldots(34)$$

There are similar theories when

$$q = \exp\left(-\pi \, \frac{z(1-x)}{z(x)}\right), \qquad z(x) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$$

is replaced by any of

$$q_1 = \exp\left(-\pi\sqrt{2} \, \frac{z_1(1-x)}{z_1(x)}\right), \qquad z_1(x) = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x\right)$$

$$q_2 = \exp\left(-\frac{2\pi}{\sqrt{3}} \, \frac{z_2(1-x)}{z_2(x)}\right), \qquad z_2(x) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)$$

$$q_3 = \exp\left(-2\pi \, \frac{z_3(1-x)}{z_3(x)}\right), \qquad z_3(x) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; x\right)$$

Ramanujan (1914):

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}}\ \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)}\right)$$

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$$q = \exp\left(-\frac{2\pi}{\sqrt{3}}\,\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)}\right)$$

J. M. Borwein and P. B. Borwein (1991)

$$x = \left(\frac{\displaystyle\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}}{\displaystyle\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}}\right)^3$$

Ramanujan's "alternative theories" of elliptic functions

Ramanujan

- pp. 257–262, second notebook
- 27 Feb 1913, second letter to G. H. Hardy
- 1914 paper "Modular equations and approximations to $1/\pi$"
  17 series for $1/\pi$

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- "There are developments of functions analogous to elliptic functions which I have not seen elsewhere..."

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Fricke (1916)

Inversion formula for ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; x)$

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K. Venkatachaliengar (1988, republished 2012)

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Berndt, Bhargava and Garvan (1995)

Proved all of the results on pp. 257–262 of Ramanujan's second notebook. (Trans. Amer. Math. Soc., 82 pages)

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C., (2009)

A unified treatment for all four theories

By Clausen's identity,

$$\begin{aligned} z^2 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)^2 &= {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; 4x(1-x)\right) \\ &= \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{x(1-x)}{16}\right)^j . \end{aligned}$$

By Clausen's identity,

$$
\begin{aligned}
z^2 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)^2 &= {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; 4x(1-x)\right) \\
&= \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{x(1-x)}{16}\right)^j .
\end{aligned}
$$

Write $Z = z^2$ and $X = x(1-x)/16$ so that

$$
Z = \sum_{j=0}^{\infty} \binom{2j}{j}^3 X^j .
$$

The series

$$Z = \sum_{j=0}^{\infty} \binom{2j}{j}^3 X^j.$$

is parameterized by

$$Z = \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^4 = \frac{1}{3} \big(4(P(q^4) - P(q)\big)$$

and

$$X = \frac{\eta_1^4 \eta_4^4}{\eta_2^4 \; Z} = \frac{\eta_1^{24} \eta_4^{24}}{\eta_2^{48}}$$

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and

$$X = \frac{\eta_1^4 \eta_4^4}{\eta_2^4 \, Z} = \frac{\eta_1^{24} \eta_4^{24}}{\eta_2^{48}}$$

where

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \qquad \eta_k = q^{k/24} \prod_{n=1}^{\infty} (1 - q^{nk}).$$

$$f(4) := \frac{4P(q^4) - P(q)}{3} \quad = \quad \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{\eta_1^4 \eta_4^4}{\eta_2^4 \; f(4)} \right)^{2j}$$

$$f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{\eta_1^4 \eta_4^4}{\eta_2^4 \, f(4)}\right)^{2j}$$

$$f(3) := \frac{3P(q^3) - P(q)}{2} = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{3j}{j} \left(\frac{\eta_1^2 \eta_3^2}{f(3)}\right)^{3j}$$

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$$f(2) := 2P(q^2) - P(q) = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{4j}{2j} \left(\frac{\eta_1^2 \eta_2^2}{f(2)} \right)^{4j}$$

$$f(4) := \frac{4P(q^4) - P(q)}{3} \;=\; \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{\eta_1^4 \eta_4^4}{\eta_2^4\, f(4)}\right)^{2j}$$

$$f(3) := \frac{3P(q^3) - P(q)}{2} \;=\; \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{3j}{j} \left(\frac{\eta_1^2 \eta_3^2}{f(3)}\right)^{3j}$$

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$$f(1) := Q(q)^{1/2} = \sum_{j=0}^{\infty} \binom{2j}{j} \binom{3j}{j} \binom{6j}{3j} \left(\frac{\eta_1^4}{f(1)}\right)^{6j}$$

$$P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}, \qquad Q(q) = 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1-q^j}$$

$$\eta_m = q^{m/24} \prod_{j=1}^{\infty} (1 - q^{mj})$$

Higher levels

$$f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{\eta_1^4 \eta_4^4}{\eta_2^4 \, f(4)} \right)^{2j}$$

$$f(5) := \frac{5P(q^5) - P(q)}{4}$$
$$= \sum_{j=0}^{\infty} \binom{2j}{j} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k} \right\} \left(\frac{\eta_1^2 \eta_5^2}{f(5)} \right)^{2j}$$

Higher levels

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$$f(5) := \frac{5P(q^5) - P(q)}{4}$$

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$$s_j = \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k}$$

$$(j+1)^2 s_{j+1} = (11j^2 + 11j + 3) s_j + j^2 s_{j-1}$$

R. Apéry: $\zeta(2) \notin \mathbb{Q}$

Rogers-Ramanujan continued fraction

$$\frac{5P(q^5) - P(q)}{4} = \sum_{j=0}^{\infty} \binom{2j}{j} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k} \right\} \left(\frac{\eta_1^2 \eta_5^2}{f(5)} \right)^{2j}$$

Rogers-Ramanujan continued fraction

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$$r = r(q) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}.$$

$$\left(\frac{\eta_1^2 \eta_5^2}{f(5)} \right)^2 = \frac{r^5(1 - 11r^5 - r^{10})}{(1 + r^{10})^2}.$$

The Rogers-Ramanujan continued fraction is an analogue of the elliptic modulus k.
More precisely, $r^5/(1 + r^{10})$ plays the role of k^2.

Summary of Lecture 3:

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The hypergeometric function $z = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ can be parameterized by the modular forms

$$z = \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^2 \quad \text{and} \quad x = \left(\frac{\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \right)^4 .$$

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$$\text{Inverse:} \quad q = \exp\left(-\pi \, \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} \right) .$$

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Other levels are open to investigation. (End of lecture 3)