Dynamics and numbers
Lecture notes for the workshop
Analysis and Dynamics in Number Theory

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Preface

These are the notes for my short course at the Abdus Salam School of Mathematical Sciences during the Second Winter Workshop on Advanced Topics in Mathematics in December 2018.

The course is concerned with the interplay between arithmetic – and in particular numeration – and dynamical systems – and in particular ergodic theory. While the theory of numeration systems and arithmetic reflects the fundamentals of our understanding of the real numbers, ergodic theory has its roots in physics and Bolzmann's ergodic hypothesis. Thus, it is perhaps at a first glance surprising that the two are in anyway related. We will however see that they are in fact intimately related, and that arithmetical phenomena can be explained via the ergodic theorem.

The course is organized in four lectures. In the first lecture, we give a brief introduction to ergodic theory completely independent from arithmetical questions. The remaining three lectures will each study a dynamical system of arithmetic origin and study the interplay between the arithmetic and dynamical properties. The first is concerned with base b-map, which encodes the base b-expansion of a real number. The second is concerned with rotations of a circle through an angle which is an irrational multiple of 2π. The third is concerned with the so-called Gauss map, which encodes the expansion of a real number as a simple continued fraction.

With all lectures, a set of exercises is supplied.

I am greatly indebted to the work of Manfred Einsiedler and Tom Ward, who wrote the excellent monograph *Ergodic theory – with a view towards number theory* [3]. Most of the material in these lectures can be found in that book; and any participant in the workshop wishing to dig deeper into these topics should definitely read it. Some background in analysis is required. A good reference is [6].

1 The Birkhoff ergodic theorem

We will describe the fundamental concepts of ergodic theory and give a proof of the von Neumann ergodic theorem as well as the Birkhoff ergodic theorem.

1.1 From abstract dynamics to ergodic theory

Let us begin with an abstract definition of a dynamical system. By a dynamical system, we mean a set X and a semigroup with identity G acting on X, i.e. for each $g \in G$, there is a map $f_g : X \to X$, and semigroup multiplication correspond to composition of maps, so that for $g_1, g_2 \in G$, $f_{g_1 g_2} = f_{g_1} \circ f_{g_2}$. Furthermore, if $e \in G$ denotes the identity, f_e is the identity map.

The setup given here is too general to do anything useful with, so we start making restrictions. First, consider the semigroup. It is natural to think of a dynamical system as the points of X moving around as time passes. Therefore, we are inclined to think as the semigroup as time. We will measure time either by $\mathbb{N}_0$ (discrete time) or by $\mathbb{R}_{\geq 0}$ (continuous time), both as additive semigroups. Most of the systems considered here are with discrete time. The reader should be aware that this is already a significant restriction. Many interesting dynamical systems – even from the point of view of number theory – are multi-parameter systems.

If the semigroup in question is in fact a group, this forces the maps by which it acts on X to be invertible. In this case, we have an action of $\mathbb{Z}$ or $\mathbb{R}$ in the one-parameter setting. Both these situations will occur in these notes.

Now for the set X (the phase space). Having restricted ourselves to one-parameter actions, we define the *orbit* of a point $x \in X$ to be the set $\{f_g(x) : g \in G\}$. In dynamics, we are typically interested in the behaviour of the orbits, but unless more structure is imposed on X, studying the dynamical system given by X and G would be a rather dull affair.

Usually, a set has a lot more structure. Furthermore, the structure should be reflected in the maps by which G acts. If X is a topological space and G acts by continuous maps (or homeomorphisms in the invertible case), this places our system in the realm of topological dynamics, where one typically studies topological properties of orbits (density, limit points, etc.). If X is a smooth manifold and G acts by smooth maps or diffeomorphisms, we are studying differentiable dynamics, and the problems mostly boil down to the study of differential equations.

Most of the systems considered here will fall into at least one of these two categories, but these structures will not be our focus. Instead, we will insist that our phase space is a probability space, and that the maps considered preserve the underlying probability measure (we will be more precise in a little while). This will lead to a study of statistical properties of the orbits.

We now restrict to the exact situation to be studied here. We will let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be some surjective map. The semigroup $\mathbb{N}_0$ acts on X by iterations of the map T, i.e. $T_n = T^n$, where T^n denotes composition of T with itself n times.

Definition 1.1. Let $(X, \mathcal{B}, \mu)$ be a probability space. A map $T : X \to X$ is said to be *measure preserving* if for any set $A \in \mathcal{B}$, $\mu(T^{-1}A) = \mu(A)$.

Likewise, if $(X, \mathcal{B})$ is a measureable space and $T : X \to X$ is a map, we say that the measure μ is T-invariant.

Measure preserving maps are the building blocks of what we will be working with. Let us begin by giving a criterion for a measure to be T-invariant.

Lemma 1.2. *A measure μ on X is T-invariant if and only if*

$$\int f d\mu = \int f \circ T d\mu$$

for all $f \in \mathcal{L}^\infty(\mu)$. In this case, the integral identity holds for all $f \in L^1(\mu)$ and indeed for $f \in L^p(\mu)$ for any $p \in [1, \infty]$.

Proof. Suppose first that the integral identity holds and let $A \in \mathcal{B}$. Let $f = \chi_A$, the indicator function of A. Then,

$$\mu(A) = \int \chi_A d\mu = \int \chi_A \circ T d\mu = \int \chi_{T^{-1}A} d\mu = \mu(T^{-1}A)$$

so that μ is T-invariant.

Suppose now that μ is T-invariant. By the above calculation, the integral identity holds true for χ_A for any $A \in \mathcal{B}$, and hence by linearity of the integral for any simple function. Let (f_n) be a sequence of simple functions increasing to some $f \in \mathcal{L}^p(\mu)$. Then, the sequence $(f_n \circ T)$ increases to $f \circ T$, and the result follows by monotone convergence. □

Applying the operation on the right hand side of the integral identity of Lemma 1.2 with $p = 2$ suggest that some operator on $L^2(\mu)$ is relevant, and indeed this is the case. We define an operator

$$U_T : L^2(\mu) \to L^2(\mu) \text{ by } U_T(f) = f \circ T.$$

It is easy to see that this is an isometry. Indeed, for $f_1, f_2 \in L^2(\mu)$,

$$\langle U_T f_1, U_T f_2 \rangle = \int f_1 \circ T \cdot \overline{f_2 \circ T} d\mu = \int f_1 \overline{f_2} d\mu = \langle f_1, f_2 \rangle.$$

The middle inequality follows by Lemma 1.2. If furthermore T is invertible, U_T becomes invertible and hence a unitary operator.

We conclude this section with the definition of ergodicity. If $(X, \mathcal{B}, \mu)$ is a probability space and $T : X \to X$ is a measure preserving transformation, it is entirely possible that the space may be decomposed into two or more T-invariant components. In an ergodic system, this cannot happen, at least not in a non-trivial way. We formalize this in a definition.

Definition 1.3. A measure preserving transformation $T : X \to X$ of a probability space $(X, \mathcal{B}, \mu)$ is *ergodic* if for any $A \in \mathcal{B}$, if $T^{-1}A = A$, then $\mu(A) \in \{0, 1\}$.

Likewise, if $T : X \to X$ is a transformation of a measureable space $(X, \mathcal{B})$, then a measure is said to be *ergodic* if T is an ergodic transformation of the probability space $(X, \mathcal{B}, \mu)$.

A set satisfying the condition that $T^{-1}A = A$ is said to be T-invariant. In the exercises, you will prove that an equivalent definition of ergodicity requires that $\mu(A) \in \{0, 1\}$ whenever $\mu(T^{-1}A \triangle A) = 0$. Here, $A \triangle B = (A \setminus B) \cup (B \setminus A)$, the symmetric set difference. Such sets are called almost T-invariant.

Yet another characterization of ergodicity, which is useful to us is in terms of T-invariant functions, i.e. functions such that $f = f \circ T$ almost everywhere.

Proposition 1.4. *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measure preserving transformation. T is ergodic if and only if any (complex valued) T-invariant function is constant almost everywhere.*

Proof. We first assume that T is ergodic and assume that f is T-invariant. Then, the real and imaginary parts of f are also T-invariant, so without loss of generality, f is real valued. For a fixed $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, let

$$A_n^k = \{x \in X : f(x) \in [\tfrac{k}{n}, \tfrac{k+1}{n})\}.$$

Note that

$$A_n^k \triangle T^{-1} A_n^k \subseteq \{x \in X : f(x) \neq f \circ T(x)\},$$

so since f is T-invariant and T is ergodic, by Exercise 1.1, $\mu(A_n^k) \in \{0, 1\}$.

Now, for any $n \in \mathbb{N}$, $X = \bigcup_{k \in \mathbb{Z}} A_n^k$, and since $\mu(X) = 1$, $\mu(A_n^k) = 1$ for exactly one value of k, $k(n)$, say. But then,

$$\mu\left(\bigcap_{n=1}^{\infty} A_n^{k(n)}\right) = 1,$$

and for x an element of this set, we must evidently have $f(x) = f \circ T(x)$.

Conversely, suppose that any T-invariant function is constant almost everywhere and let $A \in \mathcal{B}$ satisfy $\mu(T^{-1}A \triangle A) = 0$. Let $f = \chi_A$. Then, f is T-invariant, and hence constant almost everywhere. As it is an indicator function, this constant is either 0 or 1, so that $\mu(A) \in \{0, 1\}$. □

It follows that T is ergodic if and only if 1 is an eigenvalue of U_T of multiplicity 1. This is deduced in the exercises.

1.2 Ergodic theorems

Bolzmann's ergodic hypothesis states roughly that the space mean of a process is equal to its time mean with probability 1. This is the contents of the various ergodic theorems, we will now discuss. Of course, this must be made precise, and we must show that the hypothesis of a map being ergodic implies these properties. Our first result is the von Neumann ergodic theorem, which is a statement about convergence in $L^2(\mu)$. Recall that a measure preserving transformation T induces an isometric operator U_T on $L^2(\mu)$.

Theorem 1.5 (The von Neumann ergodic theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measure preserving transformation. Let P_T denote the orthogonal projection operator onto the closed subspace of invariant functions,*

$$I = \{g \in L^2(\mu) : U_T g = g\}.$$

Then, for every $f \in L^2(\mu)$*,*

$$\frac{1}{N}\sum_{n=0}^{N-1} U_T^n f \to P_T f,$$

for $N \to \infty$*, where the convergence is in* $L^2(\mu)$*.*

We give some initial remarks. For our purposes, the result is not very interesting, but it does give some hope of an interesting result. Namely, convergence in $L^2(\mu)$ is not appropriate for the problems we will be considering, but rather we would like convergence almost surely. This is our next project. Secondly, if the underlying transformation is ergodic, then by Proposition 1.4, the subspace I is one-dimensional and consists only of equivalence classes of constant functions.

Proof. We begin by identifying the orthogonal complement to I. We claim that this is the set $B = \{U_T g - g : g \in L^2(\mu)\}$. To see this, suppose first that $f \in I$, so that

$$U_T f = f.$$

Then, for any $g \in L^2(\mu)$,

$$\langle f, U_T g - g\rangle = \langle U_T f, U_T g\rangle - \langle f, g\rangle = 0,$$

since U_T is an isometry, so that $f \in B^\perp$.

Now suppose that $f \in B^\perp$. Then, for any $g \in L^2(\mu)$, $\langle U_T g, f\rangle = \langle g, f\rangle$, so that

$$\langle g, U_T^* f\rangle = \langle U_T g, f\rangle = \langle g, f\rangle,$$

so that $U_T^* f = f$, where U_T^* denotes the adjoint operator to U_T as usual. Consequently,

$$\begin{aligned}\|U_T f - f\|_2^2 &= \langle U_T f - f, U_T f - f\rangle \\ &= \|U_T f\|_2^2 - \langle f, U_T f\rangle - \langle U_T f, f\rangle + \|f\|_2^2 \\ &= 2\|f\|_2 - \langle U_T^* f, f\rangle - \langle f, U_T^* f\rangle = 0,\end{aligned}$$

so that $f = U_T f$.

It follows from orthogonal decomposition that $L^2(\mu) = I \oplus \overline{B}$, so that any $f \in L^2(\mu)$ can be written

$$f = P_T f + h,$$

where $h \in \overline{B}$. It now suffices to prove that for such an h,

$$\frac{1}{N}\sum_{n=0}^{N-1} U_T^n h \to 0$$

in $L^2(\mu)$. If $h = U_T g - g \in B$, this is trivial, as the sum telescopes and

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} U_T^n (U_T g - g) \right\|_2 = \frac{1}{N} \|U_T^N g - g\|_2 \to 0.$$

If on the other hand $h \in \overline{B} \setminus B$, we pick a sequence (h_i) in B converging to h and apply an approximation argument. We leave this for an exercise. □

We now define the ergodic averages of a function f to be

$$A_N(f) = \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n.$$

The von Neumann ergodic theorem states that in L^2, the sequence with general term $A_N(f)$ converges to $P_T f$. We would like a pointwise statement rather than a statement on convergence in L^2-norm. As a first stepping stone, we extend the result to $L^1(\mu)$.

Corollary 1.6. *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measure preserving transformation. For any $f \in L^1(\mu)$, the ergodic averages $A_N(f)$ converge in $L^1(\mu)$ to a T-invariant function $f' \in L^1(\mu)$.*

Proof. Suppose first that $g \in L^\infty(\mu) \subseteq L^2(\mu)$. Then, $A_N(g)$ converges in $L^2(\mu)$ to some $g' \in L^2(\mu)$. Furthermore, $g' \in L^\infty(\mu)$, since $\|A_N(g)\|_\infty \leq \|g\|_\infty$, so that for any $B \in \mathcal{B}$,

$$|\langle A_N(g), \chi_B \rangle \leq \|g\|_\infty \mu(B).$$

Since $A_N(g)$ converges in $L^2(\mu)$ to g', this implies that

$$|\langle g', \chi_B \rangle \leq \|g\|_\infty \mu(B),$$

for any $B \in \mathcal{B}$. But then, $\|g'\|_\infty \leq \|g\|_\infty$, and $g' \in L^\infty(\mu)$.

Now, $\|\cdot\|_1 \leq \|\cdot\|_2$ in a probability space, so $A_N(g) \to g'$ for $g \in L^\infty(\mu)$, where the convergence is in $L^1(\mu)$.

Now, let $\epsilon > 0$. Recall that $L^\infty(\mu)$ is a dense subset of $L^1(\mu)$ in the L^1-metric. Hence, if $f \in L^1(\mu)$, we may take $g \in L^\infty(\mu)$ with $\|f - g\|_1 < \epsilon$. Averaging and recalling that T preserves μ,

$$\|A_N(f) - A_N(g)\|_1 < \epsilon.$$

Since $g \in L^\infty(\mu)$, there is a $g' \in L^\infty(\mu)$ and an $N_0 \in \mathbb{N}$, such that

$$\|A_N(g) - g'\| < \epsilon,$$

whenever $N \geq N_0$. Let $N, N' \geq N_0$. Then, by the triangle inequality,

$$\begin{aligned}\|A_N(f) - A_{N'}(f)\|_1 \leq \|A_N(f) - A_N(g)\|_1 + \|A_N(g) - g'\|_1 \\ + \|g' - A_{N'}(g)\|_1 + \|A_{N'}(g) - A_{N'}(f)\|_1 < 4\epsilon.\end{aligned}$$

Hence, the sequence of ergodic averages $(A_N(f))$ is a Cauchy sequence in $L^1(\mu)$, which is complete by the Riesz–Fischer theorem. Hence, it is convergent.

Finally, to see that the limiting function in T-invariant, note that

$$\|A_N(f) \circ T - A_N(f)\|_1 \leq \frac{2}{N}\|f\|_1,$$

since the left hand side telescopes. □

For our purposes, L^1-convergence is also not sufficient. After all, we will mostly be interested in characteristic functions on measurable sets, which are evidently in all L^p-spaces. We would really like to say something about almost sure convergence. To accomplish this, we will need another two auxillary results, namely the Maximal Inequality and the Maximal Eggodic Theorem.

Proposition 1.7 (The Maximal Inequality)**.** *Let $U : L^1(\mu) \to L^1(\mu)$ be a positive linear operator with $\|U\|_{op} \leq 1$, and let $f \in L^1(\mu)$ be real valued. Define $f_0 = 0$ and*

$$f_n = \sum_{k=0}^{n-1} U^k f,$$

for $n \geq 1$. Finally, let $F_N = \max\{f_n : 0 \leq n \leq N\}$. Then, for all $N \in \mathbb{N}$,

$$\int_{\{x:F_N(x)>0\}} f d\mu \geq 0.$$

Proof. Note first that $F_N \in L^1(\mu)$. Now,

$$UF_N + f \geq Uf_n + f = f_{n+1},$$

for all $n = 0, 1, \ldots, N$, since U is a positive operator and since $F_N \geq f_n$. Hence,

$$UF_N + f \geq \max_{1 \leq n \leq N} f_n.$$

Now, let $P = \{x : F_N(x) > 0\}$ be the domain of integration. For $x \in P$, since $f_0 = 0$,

$$UF_N + f \geq \max_{1 \leq n \leq N} f_n = \max_{0 \leq n \leq N} f_n = F_N,$$

so that
$$f(x) \geq F_N(x) - UF_N(x).$$
Note now that $F_N(x) \geq 0$, so $UF_N(x) \geq 0$ as the operator U is positive. Finally, for $x \notin P$, $F_N(x) = 0$. Combining these inequalities,

$$\begin{aligned}\int_P f d\mu \geq \int_P F_N d\mu - \int_P UF_N d\mu = \int_X F_N d\mu - \int_P UF_N d\mu \\ \geq F_N d\mu - \int_X UF_N d\mu = \|F_N\|_1 - \|UF_N\|_1 \geq 0,\end{aligned}$$

where the latter inequality follows as $\|U\|_{op} \leq 1$. □

Theorem 1.8 (The Maximal Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be measure preserving. Let $g \in \mathcal{L}^1(\mu)$ be real valued and for $\alpha \in \mathbb{R}$, define*

$$E_\alpha = \left\{ x \in X : \sup_{n \geq 1} A_n(g)(x) > \alpha \right\}.$$

Then,
$$\alpha\mu(E_\alpha) \leq \int_{E_\alpha} g d\mu \leq \|g\|_1.$$
Furthermore, if $T^{-1}A = A$, then

$$\alpha\mu(E_\alpha \cap A) \leq \int_{E_\alpha \cap A} g d\mu.$$

To prove the last statement, just restrict the system to $(A, \mathcal{B}|_A, \frac{1}{\mu(A)}\mu|_A)$.

Proof. The proof is simple in view of the maximal inequality. Let $f = g - \alpha$ and $Uf = f \circ T$. Then,

$$E_\alpha = \bigcup_{N=0}^{\infty} \{x : F_N(x) \geq 0\},$$

and the maximal inequality implies that

$$\int_{E_\alpha} g d\mu - \alpha\mu(E_\alpha) = \int_{E_\alpha} g - \alpha d\mu = \int_{E_\alpha} f d\mu \geq 0.$$

□

Finally, we can prove that the convergence of the ergodic averages in fact is almost sure. This is the main contents of the Birkhoff Ergodic Theorem, which will be a key player in the coming lectures.

Theorem 1.9 (The Birkhoff Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a probability space, let $T : X \to X$ be measure preserving, and let $f \in \mathcal{L}^1(\mu)$. Then, the sequence $(A_N(f))$ converges in L^1 and almost surely to a T-invariant function $f^* \in \mathcal{L}^1(\mu)$. Furthermore,*

$$\int f^* d\mu = \int f d\mu.$$

If T is ergodic,

$$f^*(x) = \int f d\mu$$

for μ-almost every $x \in X$.

Proof. Note first that on splitting f in real and imaginary parts, it suffices to prove the result for a real valued function f. Hence, we may define

$$f^*(x) = \limsup_{N\to\infty} A_N(f)(x), \quad f_*(x) = \liminf_{N\to\infty} A_N(f)(x).$$

We must show that these two functions agree almost surely.

First, note that

$$\begin{aligned} \frac{N+1}{N} A_{N+1}(f)(x) &= \frac{N+1}{N}\left(\frac{1}{N+1}\sum_{n=0}^{N} f(T^n x)\right) \\ &= \frac{1}{N}\sum_{n=0}^{N-1} f(T^n(Tx)) + \frac{1}{N} f(x) = A_N(f)(T(x)) + \frac{1}{N} f(x). \end{aligned}$$

Considering subsequences converging to the *limsup* respectively the *liminf* on either side, we easily deduce in four steps that both f^* and f_* are T-invariant.

Now, let $\alpha > \beta$ be rational numbers and consider the set

$$E_\alpha^\beta = \{x \in X : f_*(x) < \beta, f^*(x) > \alpha\}.$$

Note that since f_* and f^* are both T-invariant, so is the set E_α^β. Let E_α be the set from the Maximal Ergodic Theorem with f in place of g. Then, clearly $E_\alpha^\beta \subseteq E_\alpha$, and the Maximal Ergodic Theorem implies that

$$\int_{E_\alpha^\beta} f d\mu \geq \alpha \mu(E_\alpha^\beta).$$

Repeating the argument with $-f$ in place of f, we similarly find that

$$\int_{E_\alpha^\beta} f d\mu \leq \beta \mu(E_\alpha^\beta),$$

so that

$$\alpha\mu(E_\alpha^\beta) \leq \beta\mu(E_\alpha^\beta).$$

Since $\beta < \alpha$, this implies that $\mu(E_\alpha^\beta) = 0$.

Now,

$$\{x \in X : f_*(x) < f^*(x)\} = \bigcup_{\substack{\alpha,\beta\in\mathbb{Q} \\ \beta<\alpha}} E_\alpha^\beta,$$

which is a countable union of null sets. Hence, $f_*(x) = f^*(x)$ for almost every $x \in X$.

To see that the almost sure limit f^* agrees with the L^1-limit from the corollary to von Neumann Ergodic Theorem, Exercise 1.4 implies that since $A_N(f)$ converges to f' in L^1, there is a subsequence $A_{N_k}(f)$, which converges almost everywhere to f'. But this limit must agree with f^* almost everywhere, and hence, the two are the same, at least almost everywhere.

Finally, the equation with the integral follows immediately on noting that

$$\int A_N(f)d\mu = \int f d\mu.$$

If T is ergodic, the only T-invariant functions are constants almost everywhere, and the final statement follows. □

Exercises

Exercises to Section 1.1

1.1 Show that ergodicity of T on $(X, \mathcal{B}, \mu)$ is equivalent to the property that for any $A \in \mathcal{B}$, if $\mu(T^{-1}A \triangle A) = 0$ then $\mu(A) \in \{0, 1\}$. (*Hint:* Show that the *limsup* set $\cap_{N=0}^\infty \cup_{n=N}^\infty T^{-n}(A)$ satisfies the invariance property of the definition and has the same measure as A).

1.2 Show that T is ergodic if and only if 1 is an eigenvalue of U_T of multiplicity 1.

Exercises to Section 1.2

1.3 Complete the proof of the von Neumann ergodic theorem by carrying out the approximation argument.

1.4 Let $(X, \mathcal{B}, \mu)$ be a measure space. Show that if (f_n) is a sequence in $L^p(\mu)$ $(1 \leq p < \infty)$, which converges to $f \in L^p(\mu)$, then there is a subsequence (f_{n_k}) which converges to f pointwise almost surely.

(*Hint:* Take a subsequence (f_{n_k}) with $\|f_{n_k} - f\|_p^p < k^{-2-p}$ and show that

$$\mu\left\{x \in X : |f_{n_k}(x) - f(x)| > \tfrac{1}{k}\right\} < \tfrac{1}{k^2}.$$

Now apply the Borel–Cantelli lemma.)

1.5 Deduce the following corollary of the Birkhoff ergodic theorem:

$(X, \mathcal{B}, \mu)$ be a probability space, let $T : X \to X$ be measure preserving, and let $E \in \mathcal{B}$. Then almost every $x \in E$ returns to E infinitely often, i.e. there is a set $F \subseteq E$ with $\mu(F) = \mu(E)$, such that for each $x \in F$, there is an increasing sequence of natural numbers (n_k), such that $T^{n_k} \in E$. This is the Poincaré recurrence theorem, which can also be proved independently of the ergodic theorem.

2 Markov shifts and the base b-map

A particularly simple example of an ergodic system is the base-b map, which encodes the digit distribution of the initial point in an orbit. We will prove the ergodicity (and more) of this map and continue with some arithmetic consequences; in particular an ergodic proof of Émile Borel's classical result that almost all numbers are absolutely normal.

2.1 Base b-maps, coin flips and shift maps

It is well known that given a real number $x \in [0,1)$ and an integer $b \in \mathbb{Z}$ with $b \geq 2$, we can express x in the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{b^i}, \quad a_i \in \{0, 1, \ldots, b-1\}.$$

As it turns out, there is an underlying dynamical system to this expansion. To see this, note that we may read off the first digit a_1 by splitting up the unit interval $[0,1)$ into b disjoint intervals of equal size, namely $[\frac{j}{b}, \frac{j+1}{b})$. If $x \in [\frac{j}{b}, \frac{j+1}{b})$, then $a_1 = j$. Thus, we define the function $a : [0,1) \to \{0, 1, \ldots, b-1\}$ by,

$$a(x) = j \quad \text{for } x \in [\tfrac{j}{b}, \tfrac{j+1}{b}).$$

Finding a_2 requires a little more work. For a real number x, let $[x]$ denote the integer part of x (or the floor function), and let $\{x\}$ denote the fractional part of x, i.e. $\{x\} = x - [x]$. Multiplying x by b, we find that

$$bx = b \sum_{i=1}^{\infty} \frac{a_i}{b^i} = a_1 + \sum_{i=1}^{\infty} \frac{a_{i+1}}{b^i}.$$

Taking fractional parts,

$$\{bx\} = \sum_{i=1}^{\infty} \frac{a_{i+1}}{b^i},$$

and thus $a_2 = a(\{bx\})$. We may of course repeat the procedure to find a_3, a_4 and so on.

To put this into the dynamical context, define the map $T_b : [0,1) \to [0,1)$ by $T_b(x) = \{bx\}$. In this way, we find that

$$x = \sum_{i=1}^{\infty} \frac{a(T_b^{i-1}x)}{b^i},$$

so that a base b-expansion of x is fully determined by the orbit of x under T_b together with the test function a. The map T_b acts as a *shift map* on the digits of x: it removes the first digit and shifts the rest up.

When we say *a* base b-expansion and not *the* base b-expansion, this is entirely on purpose. Indeed, the base b-expansion of a real number is not always unique, as is easily seen by considering the decimal expansion of $1/10$, for which

$$\frac{1}{10} = \sum_{i=2}^{\infty} \frac{9}{10^i}.$$

Of course, one could consider the shift map on the full set of sequences, i.e. one could ignore the fact that a number may have more than one base b-expansion. From this point of view, we consider a finite alphabet with b letters, $\Lambda = \{0, 1, \ldots, b-1\}$, and as the phase space of our dynamical system we consider the set of sequences with elements from Λ, namely $X = \Lambda^{\mathbb{N}}$. Equipping the space with the shift map $\sigma : X \to X$, given by

$$\sigma\left((a_i)_{i=1}^{\infty}\right) = (a_{i+1})_{i=1}^{\infty},$$

we obtain a dynamical system in the sense first defined. Indeed, we have a map and an action of the semigroup $\mathbb{N}$.

To get sufficient structure on the shift space X to apply any additional theory, we do the following. First, if we equip Λ with the discrete topology, it is entirely natural to put the product topology on X, so the dynamical system is certainly a topological dynamical system. By Tychonoff's Theorem, the space is compact, and in Exercise 2.1, you will show that in fact this topology is metrizable in a very concrete way.

Our objective in these notes is to study ergodic theory, so we will now construct a family of measures on shift spaces. Let $\overline{\mu}$ be some measure on $\Lambda = \{0, 1, \ldots, b-1\}$, i.e. any function from Λ to $[0,1]$, such that $\sum_{k \in \Lambda} \overline{\mu}(k) = 1$. On $X = \Lambda^{\mathbb{N}}$, this induces the (infinite) product measure $\mu = \prod_{i \in \mathbb{N}} \overline{\mu}$, called a *Bernoulli measure.*

A very concrete interpretation is the following. Let $b = 2$, so that $\Lambda = \{0, 1\}$, and let $\overline{\mu}$ be the uniform probability distribution on $\{0, 1\}$, which assigns measure $\frac{1}{2}$ to each element. This probability space corresponds to flipping a fair coin. The measure on the product space $\Lambda^{\mathbb{N}}$ corresponds to repeatedly flipping this coin an infinite number of times.

It is thus not at all surprising that the shift map is ergodic for this particular system. Indeed, the Strong Law of Large Numbers say that for a process of independent and identically distributed random variables,

$$\frac{1}{N}\sum_{n=1}^{N} X_n \to \mathbb{E}(X_1)$$

almost surely. A Random variable is nothing but an integrable function on $\{0, 1\}$ (so any function will do). Reading this in terms of the shift map, the left hand side is just the ergodic averages of some such function, while the right hand side is the integral of this function with respect to μ. We have recovered the Birkhoff ergodic theorem for this particular system.

Of course, there is nothing in our setup requiring that the coin is fair! It may just be that the coin comes up heads 3/4 of the time, and we would get a different distribution. However, the Strong Law of Large Numbers would remain valid. We will return to this issue, but for now let ud summarize the discussion on shift spaces in a defintion.

Definition 2.1. Let Λ be a finite set and let $X = \Lambda^{\mathbb{N}}$ be the set of all (one-sided) sequences with elements from Λ. Let $\overline{\mu}$ be a probability measure on Λ and let $\mu = \prod_{n\in\mathbb{N}} \overline{\mu}$ be the infinite product measure on $\mathbb{N}$. Note that this is certainly a Borel probability measure. The probability space $(X, \mathcal{B}, \mu)$ together with the shift map $\sigma : X \to X$ is called a *Bernoulli shift*.

2.2 Isomorphic systems

It appears that proving that a Bernoulli shift is ergodic should be a simple matter in view of the above remarks on the Strong Law of Large Numbers. However, our objective is to study numbers, and in particular the base b-expansions of real numbers, and the base b-map T_b is not a Bernoulli shift, due to the non-uniqueness of base b-expansions.

We now show that from the point of view of ergodic theory, the base b-map on $[0, 1)$ with Lebesgue measure and the Bernoulli shift on b letters with the uniform measure are indistinguishable. For this, we need the notion of isomorphic dynamical systems. A map $\phi : X \to Y$ between measure spaces $(X, \mathcal{B}_X, \mu)$ and $(Y, \mathcal{B}_Y, \nu)$ is measure preserving if for any $A \in \mathcal{B}_Y$, we have $phi^{-1}(A) \in \mathcal{B}_X$ with $\mu(\phi^{-1}(A)) = \nu(A)$.

Definition 2.2. Let $(X, \mathcal{B}_X, \mu)$ and $(Y, \mathcal{B}_Y, \nu)$ be probability spaces and let $T : X \to X$ and $S : Y \to Y$ be measure preserving transformations of X and Y respectively. The two measure preserving systems are said to be *isomorphic* if there exist sets $X' \subseteq X$ and $Y' \subseteq Y$ with $\mu(X') = \nu(Y') = 1$ such that $TX' \subseteq X'$ and $SY' \subset Y'$ and an invertible, measure preserving map $\phi : X' \to Y'$ such that

$$\phi \circ T(x) = S \circ \phi(x)$$

for all $x \in X'$.

Note that if two systems are isomorphic and one is ergodic, then immediately so is the other one.

Proposition 2.3. *Let $b \in \mathbb{Z}$ with $b \geq 2$ and let λ denote the Lebesgue measure restricted to $[0, 1)$. The system $([0, 1), \mathcal{B}, \lambda)$ with the map $T_b : [0, 1) \to [0, 1)$ is isomorphic to the Bernoulli shift $\{0, 1, \ldots, b-1\}^{\mathbb{N}}$ based on the uniform probability distribution.*

Proof. We first observe, that the σ-algebra in the shift space $\{0, 1, \ldots, b-1\}^{\mathbb{N}}$ is generated by cylinder sets, i.e. sets of the form

$$C(c_1, c_2, \ldots, c_M) = \{(a_i)_{i=1}^{\infty} : a_i = c_i \text{ for } i = 1, 2, \ldots, M\},$$

where $M \in \mathbb{N}$ and $c_1, c_2, \ldots, c_M \in \{0, 1, \ldots, b-1\}$ are fixed. Also, observe that the σ-algebra $\mathcal{B}$ in $[0, 1)$ is generated by the b-adic intervals,

$$\left[\frac{j}{b^k}, \frac{j+1}{b^k}\right),$$

where $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, b^k - 1\}$ are fixed. Hence, it suffices to provide an invertible map between sets of this type, which preserves the measure.

The map ϕ is the obvious one, which takes the base b-expansion generated in the beginning of the chapter and maps to the sequence of digits. The domain of ϕ will thus be $[0, 1)$. The image is not all of $\{0, 1, \ldots, b-1\}^{\mathbb{N}}$, as we are only hitting one particular base b-expansion whenever there are more. This is easily dealt with, as we know (see e.g. [4]) that the only way in which a number can have more than one base b-expansion is if it has expansions ending in an infinite number of 0's and in an infinite number of $(b-1)$'s. This happens only for countably many numbers, so $\{0, 1, \ldots, b-1)^{\mathbb{N}} \setminus \phi([0, 1))$ is countable. Since the Bernoulli measure is evidently non-atomic, a countable set has measure 0, and we now have a map between full subsets of the interval $[0, 1)$ and the Bernoulli shift. It is evident from the construction that the map intertwines the shift map and the base b-map, T_b.

It remains for us to prove that the map preserves the measure. Let $M \in \mathbb{N}$ and $c_1, c_2, \dots, c_M \in \{0, 1, \dots, b-1\}$ be fixed and consider the preimage of the cylinder set $C(c_1, c_2, \dots, c_M)$ under ϕ. This is the set

$$\left\{x \in [0,1) : x = \sum_{i=1}^{M} \frac{c_i}{b^i} + \sum_{i=m+1}^{\infty} \frac{a_i}{b^i}, a_i \in \{0, 1, \dots, b-1\}\right\},$$

an interval of length b^{-M}. But this is also the Bernoulli measure of the original cylinder $C(c_1, c_2, \dots, c_M)$. This completes the proof. □

With the above proposition in place, in order to prove that T_b is ergodic with respect to the Lebesgue measure, it suffices to prove that Bernoulli shifts are ergodic. We do this now.

Theorem 2.4. *Bernoulli shifts are ergodic.*

Proof. The cylinder sets generate the σ-algebra $\mathcal{B}$. This means that for any set $B \in \mathcal{B}$ and any $\epsilon > 0$, there is an $N \in \mathbb{N}$ and a set $F \subseteq \Lambda^N$, such that the set

$$A = \{(a_i)_{i=1}^{\infty} : (a_1, a_2, \dots, a_N) \in F\}$$

satisfies that $\mu(A \triangle B) < \epsilon$. Let $\epsilon \in (0, 1)$, let B be σ-invariant and pick such an A.

Take an A of this form. Then, for $M > N$,

$$\sigma^{-M} A = \{(a_i)_{i=1}^{\infty} : (a_{M+1}, a_{M+2}, \dots, a_{M+N}) \in F\}.$$

Note that the entries of the sequences in $\sigma^{-M} A$ and A on which restrictions are imposed by the structure of A are at a disjoint set of indices. Hence, since the measure is a product measure,

$$\mu(\sigma^{-M} A \setminus A) = \mu(\sigma^{-M} A \cap (X \setminus A)) = \mu(\sigma^{-M} A)\mu X \setminus A) = \mu(A)\mu(X \setminus A),$$

since σ is measure preserving.

Since B is assumed to be σ-invariant, $\mu(B \triangle \sigma^{-1} B) = 0$. Hence,

$$\mu(\sigma^{-M} A \triangle B) = \mu(\sigma^{-M} A \triangle \sigma^{-M} B) = \mu(A \triangle B) < \epsilon,$$

so that

$$\mu(\sigma^{-M} A \triangle A) < 2\epsilon.$$

But then,

$$\mu(\sigma^{-M} A \setminus A) \le \mu(A \setminus \sigma^{-M} A) + \mu(\sigma^{-M} A \triangle A) = \mu(\sigma^{-M} A \triangle A) < 2\epsilon.$$

Now, finally

$$\begin{aligned}\mu(B)\mu(X\setminus B) &< (\mu(A)+\epsilon)(\mu(X\setminus A)+\epsilon)\\ &= \mu(A)\mu(X\setminus A)+\epsilon\mu(A)+\epsilon\mu(X\setminus A)+\epsilon^2\\ &< \mu(\sigma^{-M}A\setminus A)+3\epsilon < 5\epsilon.\end{aligned}$$

Since ϵ is arbitrary, this shows that $\mu(B)\mu(X\setminus B) =$, so that $\mu(B)\in\{0,1\}$. □

Of course, Theorem 2.4 opens a whole new can of works! It says that *any* Bernoulli shift is ergodic. Thus, if we start with a different measure on $\{0,1,\dots,b-2\}$, we obtain an entirely different ergodic system (the resulting measures on the shift space are clearly different). It is reasonable to ask whether such a new measure gives rise to an ergodic measure on $[0,1)$. The answer is yes! Suppose we have two measurable spaces $(X,\mathcal{B}_X)$ and $(Y,\mathcal{B}_Y)$ with a measurable bijection $\phi: X\to Y$ which intertwines automorphisms $T: X\to X$ and $S: Y\to Y$ in the sense used to define isomorphic systems. Suppose further that T preserves a measure μ on X. We may then define a measure ν on Y by letting

$$\nu(A)=\mu(\phi^{-1}(A)),$$

for any $A\in\mathcal{B}_Y$. Thus, there are infinitely many ergodic measures for the map T_b. We now examine this fact further.

Let us digress a little into functional analysis. Riesz' Representation Theorem provides a correspondence between positive measures and positive linear functionals on $C_c(X)$, the space of continuous functions with compact support on X: Two measures μ and ν are equal as measures if

$$\int f\,d\mu=\int f\,d\nu$$

for all $f\in C_c(X)$. As such, the set of measures is a subset of the dual space $C_c(X)^*$ to the Banach space $C_c(X)$ endowed with the weak*-topology.

Now, the set of probability measures is a subset of the unit ball (in fact of the unit sphere) of $C_c(X)^*$. The unit ball is compact by the Banach–Alaoglu Theorem, and it is easily checked that being invariant for a map $T: X\to X$ is a closed and convex property. In other words, the set of preserved measures form a compact and convex subset of $C_c(X)^*$.

Working a little harder, one finds that the extremal points of this set are exactly the ergodic measures for T. One might think that such a set is a simple thing to study, but this is far from the case. In exercise 2.2, you will construct a compact and convex subset of a Banach space, inside which the extremal points are dense. This is generically the case for dynamical systems. The set of ergodic measures is very badly behaved.

Before proceeding, let us mention that the set of preserved measures may be the empty set. This will not happen to us, and the existence of preserved (and hence ergodic) measures is guaranteed, if the space X considered is a compact metric space (for us the circle), and the map T is continuous.

2.3 Borel's theorem on normal numbers

Let us finally derive a result from number theory from the work done so far. Let $b \geq 2$ be an integer. A real number $x \in [0, 1)$ is said to be normal to base b if all finite blocks of digits occur with the expected frequency in a base b expansion of x, i.e. if for all $N \in \mathbb{N}$ and all $(c_1, c_2, \ldots, c_M) \in \{0, 1, \ldots, b-1\}^M$,

$$\lim_{N\to\infty} \frac{\#\{n \leq N : (a_{n+1}, a_{n+2}, \ldots, a_{n+M})}{N} = (c_1, c_2, \ldots, c_M)\} \ldots N = b^{-M}. \tag{2.1}$$

A number is *normal* if it is normal to any integer base $b \geq 2$.

The following result is due to Émile Borel [2].

Theorem 2.5. *Lebesgue almost all real numbers in* $[0, 1)$ *are normal.*

Proof. First, fix a base b and a block of digits, $(c_1, c_2, \ldots, c_M) \in \{0, 1, \ldots, b-1\}^M$. Define the set

$$A = \left\{ x = \sum_{i=1}^{\infty} \frac{a_i}{b^i} : (a_1, a_2, \ldots, a_M) = (c_1, c_2, \ldots, c_M) \right\}.$$

Note that this is an interval of length B^{-M}. Let $f = \chi_A$, the characteristic function of this interval. Clearly, this is a function in $L^1(\lambda)$, and since the map $T_b : [0, 1) \to [0, 1)$ is ergodic with respect to λ, the Birkhoff ergodic theorem implies that

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T_b^n x) \to \int f d\lambda$$

for λ-almost every $x \in [0, 1)$. Now, note that $f(T_b^n) = 1$ if and only if $(a_{n+1}, a_{n+2}, \ldots, a_{n+M}) = (c_1, c_2, \ldots, c_M)$ and $f(T_b^n x) = 0$ otherwise; and clearly $\int f d\lambda = b^{-M}$. Hence, the above equation is nothing but (2.1) in disguise, so this particular block of digits occurs with the expected frequency.

Since the block of digits was arbitrary, and there are only countably many possible blocks, the set of numbers for which any one of these limits does not exist or has a different value than the expected is also a null set. This shows that almost all numbers are normal to base b.

Finally, to prove normality of almost all numbers, note that the set of numbers which fail to be normal to base b is a null set. As there are only countably many bases, the union of all of these sets is itself a null set. Everything in its complement is normal, and the complement is full with respect to Lebesgue. □

We have only considered numbers between 0 and 1 at this point, but since the requirement to be normal is an asymptotic one, anything taking place in the integer part of a number will have no effect on the combinatorially defined normality. The result makes it a reasonable question whether any nice and well-known number, such as π, e, $\sqrt{2}$ or $\log 2$, can be shown to be normal. These are all open (and seemingly very difficult) problems! It is conjectured that algebraic numbers which are not irrational are normal, but very little is known about this. Classical transcendental constants such as π are also well out of reach.

One can certainly construct normal numbers to a specific base. A famous example is Champernowne's number, which in base 10 is given by

$$0.1234567891011\ldots$$

Numbers normal to any base can also be constructed, though involves a lot more technique. In recent years, a construction has been given, which generates the n'th binary digit of such a number in close to quadratic time [1]. The number does not however resemble anything from other parts of mathematics.

Exercises

Exercises to Section 2.1

2.1 Let Λ be a finite set. Show that on $X = \Lambda^{\mathbb{N}}$, the function $d : X \times X \to \mathbb{R}_{\geq 0}$ given by

$$d\left((a_i)_{i=1}^{\infty}, (b_i)_{i=1}^{\infty}\right) = \begin{cases} 0 & \text{if } (a_i)_{i=1}^{\infty} = (b_i)_{i=1}, \\ 2^{-\min\{i \in \mathbb{N} : a_i \neq b_i\}} & \text{otherwise,} \end{cases}$$

defines a metric on X (Here, the minimum of the empty set $\emptyset$ is set to be equal to 0). Show that this metric induces the product topology of the discrete topology on Λ.

Exercises to Section 2.2

2.2 In this exercise, you will construct the so-called Poulsen simplex, see also [7]. The construction takes place in $\ell^2(\mathbb{R})$, but simplexes of this form are the typical shape of the set of preserved measures for dynamical systems.

First some notation. For a convex set S, let $E(S)$ denote the extremal points in S. Let $e_j \in \ell^2(\mathbb{R})$ denote the sequence which has 1 as it's j'th entry and 0's everywhere else. Finally, let E_n denote the subspace spanned by $\{e_1, e_2, \ldots, e_n\}$

(a) Construct a sequence of simplexes S_n with the following properties:

(1) $S_n \subseteq E_n$ for every n.

(2) $S_n \subseteq S_m$ and $E(S_n) \subseteq E(S_m)$ for $m > n$.

(3) $P_n S_m = S_n$ for $m > n$.

(4) For any $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that the maximal distance from any $x \in S_n$ to $E(S_n)$ is at most ϵ.

(*Hint:* Start with a line segment of length 2^{-1} in E_1. Then pick finitely many points such that any point in the segment is within a distance of 2^{-2} of these points. Finally, stick on an orthogonal direction on each of these points – one direction for each point – and take convex hulls successively. Repeat!)

(b) With S_n constructed, let

$$T_n = P_n^{-1}(S_n) \quad \text{and} S = \bigcap_{n=1}^{\infty} T_n.$$

Show that

(1) $T_n \supseteq E_n$ for $n < m$.

(2) $P_n T_m = S_n$ for $n < m$.

(3) $P_n S = S_n$ for all n.

(4) The set $\bigcup_{n=1}^{\infty} E(S_n)$ is dense in S.

(c) Show that $E(S)$ is dense inside S. (*Hint:* It is enough to prove that $E(S_n) \subseteq E(S)$).

(d) Show that S is a simplex, i.e. that any set of the form

$$A = S \cap (qS - a), \quad q > 0$$

containing at least 2 points is itself a homothetic copy of S, i.e.

$$A = rS + b, \quad r > 0.$$

Exercises to Section 2.3

2.3 In this exercise, we look at a particular instance of the so-called β-expansions of Parry [5] and Rényi [8]. Most questions are open-ended, and you are encouraged to play around with the example.

Let $\phi = (1+\sqrt{5})/2$, the Golden Ratio and consider the map $T_\phi : [0,1) \to [0,1)$. Show that the dynamical system is conjugate so the subspace of shift space on two elements given by

$$\Sigma_\phi = \left\{ (a_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : a_i = 1 \Rightarrow a_{i+1} = 0 \right\}.$$

Does this have a meaning in the representation of real numbers on the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\phi^i}, \quad a_i \in \{0,1\}?$$

Can you say anything about the ergodic properties of the map? What does it mean for a number to be normal to base ϕ?

The shift space in this exercise is a so-called sub-shift of finite type. These are defined by prohibiting a finite number of sub-words from the full shift. In this case, no sequence can contain the subword 11. Feel free to play around with finding a condition on $\beta > 1$ implying that $T_\beta : [0,1) \to [0,1)$ is conjugate to a sub-shift of finite type.