Dynamics and numbers

Lecture notes for the workshop
Analysis and Dynamics in Number Theory

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3 Rotations of the circle and unique ergodicity

At the other extreme of the possible behaviours of circle maps from the Markov shifts of Lecture 2, we find rotations of the circle. These admit only a single preserved probability measure. We will give a detailed description of irrational rotations of the circle and show how the behaviour of orbits is completely governed by the arithmetic properties of the angle of rotation.

3.1 Unique ergodicity

A slightly annoying thing about the Birkhoff ergodic theorem is the fact that the convergence is only almost sure. Of course, the phenomenon is very real. If we had convergence for all initial values of x, we could use the methods of the last lecture to prove that all real numbers were normal, which is quite clearly not the case. Nonetheless, for some systems the convergence holds true for all values of x, as we shall now see.

The orbits for which the convergence statement does not hold stem from other points in the set of preserved measures, at least in the case when X is compact metric and T is continuous. We prove this.

Proposition 3.1. *Let X be a compact metric space and let $T : X \to X$. Let (ν_n) be a sequence of probability measures on X and let $T_*^j \nu_n(A) = \nu_n(T^{-j}A)$ denote the pushforward measure of ν_n under T^j. Then, any weak*-limit point of the sequence*

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} T_*^j \nu_n$$

is a T-invariant probability measure.

Proof. Let μ be any weak*-limit point of (μ_n). We must check that for any continuous function $f : X \to \mathbb{R}$,

$$\int f \circ T d\mu = \int f d\mu.$$

Take a subsequence $\mu_{n(j)}$ which converges to μ. Then,

$$\begin{aligned}\left|\int f \circ T d\mu_{n(j)} - \int f d\mu_{n(j)}\right| &= \frac{1}{n(j)}\left|\int \sum_{i=0}^{n(j)-1} (f \circ T^{i+1} - f \circ T^i) d\nu_{n(j)}\right| \\ &= \frac{1}{n(j)}\left|\int (f \circ T^{n(j)} - f) d\nu_{n(j)}\right| \\ &\leq \frac{2\|f\|_\infty}{n(j)}.\end{aligned}$$

Since the latter tends to 0 as j increases, we are done. □

Note that any T-invariant measure can be written as a convex combination of ergodic measures (by the Krein–Millman theorem), and that such a weak limit must exist (by the Banach–Alaoglu theorem). Of course, these results do not assume compactness of X, and we should explain this. In the non-compact setting, one could easily have a situation, where a sequence of measures such as the one in the Proposition has a support which moves to infinity, i.e. which leaves any compact subset as n increases. Any weak*-limit would then be the zero-functional, which most certainly does not correspond to a probability measure. It is however a linear functional, so there is no contradiction with Banach–Alaoglu. The phenomenon is called escape of mass.

With this result in hand, it would seem that the measure obtained by letting $\nu_n = \delta_x$ would give an invariant measure more suitable for the study of the particular orbit of x, and this is indeed the case.

Now, it could just be that the convex set of preserved measures consists of a single point. If this is the case, any sequence of measures as in Proposition 3.1 would have to be convergent with that exact limit measure. In this case, the only extremal point of the set of preserved measures would be that particular measure, so the system will have to be ergodic.

Definition 3.2. Let X be a compact metric space and let $T : X \to X$ be continuous. Then, T is said to be *uniquely ergodic* if there is only one T-invariant Borel measure on X.

As it turns out, this is what it takes for us to be able to remove the almost everywhere clause from the Birkhoff ergodic theorem

Theorem 3.3. *Let X be a metric space and let $T : X \to X$ be continuous. T is uniquely ergodic if and only if for every $f \in C(X)$ and every $x \in X$,*

$$A_N(f)(x) \to C_f,$$

where C_f is is a constant independent of x.

If one works a little harder, it is possible to show that the convergence is in fact uniform over X. We stick with the weaker statement.

Proof. Suppose first that T is uniquely ergodic and let $x \in X$. In Proposition 3.1, let $\nu_n = \delta_x$, the point probability measure with support $\{x\}$. The probability measures μ_N of that proposition now become

$$\mu_N = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n x},$$

and the sequence converges to the unique T-invariant measure μ on X. Now, for $f \in C(X)$,

$$A_N(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f d\mu_N \to \int f d\mu.$$

Conversely, let μ be a T-invariant probability measure and suppose that the ergodic averages converge for every $f \in C(X)$ and every $x \in X$. Note that since the measure μ is preserved by T, for any $N \in \mathbb{N}$,

$$\int f d\mu = \int \frac{1}{N} \sum_{n=0}^{\infty} f(T^n x) d\mu = \int A_N(f) d\mu.$$

Since a continuous function on a compact set is bounded, we may without further ado apply the dominated convergence theorem. By that theorem, for any $f \in C(X)$,

$$\int f d\mu = \int \lim_{N \to \infty} A_N(f) d\mu = \int C_f d\mu = C_f.$$

But this completely characterizes a measure on X, so there can be only one. □

In the next section, we will show that a uniquely ergodic system does in fact exist, and that it is even of interest to a number theorist.

3.2 Rotations

A very natural dynamical system is a rotation of the circle. Here, one takes a point on the unit circle

$$\mathbb{S} = \{z = e^{2\pi i x} : x \in \mathbb{R}\}$$

and rotates it through an angle $2\pi\alpha$, i.e. we consider the map $R_\alpha : \mathbb{S} \to \mathbb{S}$ given by

$$R_\alpha(e^{2\pi i x}) \to e^{2\pi i x} e^{2\pi i \alpha} = e^{2\pi i (x+\alpha)}.$$

Alternatively (and in a little cleaner notation) we could identify the unit circle $\mathbb{S}$ with the unit interval $[0, 1)$ in which we identify the endpoints. Then, the rotation becomes (by abuse of notation denoted in the same way) $R_\alpha : [0, 1) \to [0, 1)$ given by

$$R_\alpha(x) = \{x + \alpha\}.$$

If $\alpha \in \mathbb{Q}$, this is a rather boring map. Indeed, if $\alpha = p/q$, the orbit of any point returns to itself after q iterations. If on the other hand $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we have what is known as an irrational rotation, which is a much more interesting system.

Proposition 3.4. *An irrational rotation of a circle is uniquely ergodic.*

Proof. We argue by Theorem 3.3. Consider first the function $f(x) = e^{2\pi i k x}$, where $k \in \mathbb{Z}$ is fixed. Then,

$$A_N(f) = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i k (x+n\alpha)}.$$

If $k = 0$, this is equal to 1. If on the other hand $k \neq 0$, this is a geometric sum, and

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i k (x+n\alpha)} = \frac{1}{N} e^{2\pi i k x} \frac{e^{2\pi i N k \alpha} - 1}{e^{2\pi i k \alpha} - 1},$$

which tends to 0 as N tends to infinity. Now, note that

$$\int f d\lambda = \int_0^1 e^{2\pi i k x} dx = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

In other words, for these particular functions,

$$A_N(f) \to \int f d\lambda,$$

regardless of base point x.

By linearity, the same convergence holds true for linear combinations of these functions, i.e. for trigonometric polynomials. By the Stone–Weierstrass theorem, the set of these is dense in $C(X)$, and hence it only requires a simple approximation argument to see that the result is true for all functions in $C(X)$. Hence, by Theorem 3.3, the rotation is uniquely ergodic. $\square$

Note that for this system, since it is uniquely ergodic, we need not worry about the starting point for the orbit. The limit is the same regardless of this choice. A really funny application of this is to be found in Exercise 3.1

3.3 Further results on rotations

A rotation R_α clearly must encode some arithmetic properties of the irrational number α. We proceed with an informal discussion of some of these.

The first thing, which is somewhat surprising, is the so-called Three Distances Theorem. Note that because of unique ergodicity, it does not matter which starting point we take for our dynamics, so we may as well start with 0.

Theorem 3.5. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Consider the points $\{0, R_\alpha(0), \dots, R_\alpha^N(0)\}$. These partition the circle into $N+1$ arcs. The set of lengths of these arcs has cardinality at most 3 for any N.*

This is surprising, since one can think of ergodicity as an instance of chaos: Orbits are dense and in fact uniformly distributed. Nevertheless, by this theorem, there is an awful lot of structure to the orbits. One can relatively easily see why such a result might be true by drawing the orbit of 0 for a specific α. A particularly nice exposition is found in [11], where it is also shown how to reconstruct the simple continued fraction expansion of α from the orbits of the associated rotation.

Another result on rotations looks a little like Diophantine approximation, but where one approximates with the orbit of a rotation rather than with rational numbers. Very much like the corollary to Dirichlet's Theorem, covered in the lectures of Mumtaz Hussain, Kim [6] showed that for almost all $x \in [0, 1)$, there are infinitely many $n \in \mathbb{N}$, such that

$$|R_\alpha^n(0) - x| < \frac{1}{n}.$$

It was later shown by Bugeaud, Harrap, Kristensen and Velani [3] that the analogue of badly approximable numbers also exist in this setting. We will not prove this here, but will mention that the proof combines many of the ideas of this Workshop: continued fractions and non-typical orbits of base b-maps (or rather something a little more flexible) and quite a bit of measure theory.

Exercises

Exercises to all sections

3.1 A set $A \subseteq \mathbb{N}$ is said to have density $d(A)$ if

$$d(A) = \lim_{N\to\infty} \frac{\#\{a \in A : a \leq N\}}{N}$$

exists. Let $k \in \{0, 1, \ldots, 9\}$, and consider the set

$$A = \{n \in \mathbb{N} : \text{the leading digit of } 2^n \text{ is } k\}.$$

Show that the density of A is $\log_{10}((k+1)/k)$.

3.2 Extend the result on unique ergodicity of irrational rotations to the following theorem:

Theorem 3.6. *Let X be a compact metric group with Haar measure μ_X and let $R_g(x) = gx$ be the rotation of X by a fixed element $g \in X$. The following are equivalent*

(a) R_g is uniquely ergodic with invariant measure μ_X.

(b) R_g is ergodic with respect to μ_X.

(c) The subgroup $\{g^n\}_{n\in\mathbb{Z}}$ generated by g is dense in X.

(*Hint:* Show that (a) implies (b), which implies (c) (argue by contradiction), which in turn implies (a) (using that (c) implies that X is abelian, so that any R_g-invariant measure is invariant under translation by a dense group. Now show that such a measure must be translation invariant under translation by any element and conclude)).

3.3 Extend the list in the preceding exercise by proving that (b) is equivalent to the statement that

(a) X is abelian and $\chi(g) \neq 1$ for any non-trivial character $\chi \in \hat{X}$.

4 The Gauss map and the Gauss measure

The Gauss map provides a dynamical description of the simple continued fraction algorithm. It has the nice feature of being ergodic with respect to a specific measure, the Gauss measure. We will outline a proof of this and deduce a number of classical results in the metric theory of continued fractions from the Birkhoff ergodic theorem.

4.1 The map and the measure

We first summarize some properties about continued fractions, which are by now known from the courses of Karl Dilcher and Mumtaz Hussain.

Proposition 4.1. *The continued fraction algorithm has the following properties:*

1. *The convergents may be calculated from the following recurrence formulae: Let $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$ and $q_0 = 1$. For any $n \geq 1$,*

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}.$$

   Consequently, $q_n \geq 2^{(n-1)/2}$.

2. *For any $n \geq 0$*

$$q_n p_{n-1} - q_{n-1} p_n = (-1)^n,$$

   and for any $n \geq 1$,

$$q_n p_{n-2} - q_{n-2} p_n = (-1)^{n-1} a_n$$

3. *For an irrational number x, $x - p_n/q_n$ is positive if and only if n is even.*

4. *Any real irrational number x has an expansion as a continued fraction. The sequence of convergents of x converges to x, with the even (resp. odd) order convergents forming a strictly increasing (resp. decreasing) sequence. This expansion is unique, and we write $x = [a_0; a_1, \dots]$.*

5. *Given a sequence $\{a_n\}_{n=0}^{\infty}$ with $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for $i \geq 1$, the sequence $[a_0; a_1, \dots, a_n]$ converges to a number having the sequence $\{a_n\}$ as its sequence of partial quotients.*

6. *The convergents satisfy*

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.$$

We consider $x \in [0, 1)$ and formalise the continued fraction algorithm in the form of a self-mapping of the unit interval.

Definition 4.2. The *Gauss map* $T : [0, 1) \to [0, 1)$ is defined by

$$Tx = \begin{cases} \{1/x\} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Note that this map has the effect of shifting the sequence of partial quotients, just like the base b-map for the b-adic expansion. To extract the first partial quotient, we use the map

$$a(x) = \begin{cases} [1/x] & \text{for } x \neq 0 \\ \infty & \text{for } x = 0. \end{cases} \tag{4.1}$$

We now see that

$$a_n(x) = a(T^{n-1}x), \tag{4.2}$$

where $a_n(x)$ denotes the n'th partial quotient in the continued fraction expansion of x, so iterates of the Gauss map are the natural object to study.

Having established that the Gauss map encodes the behaviour of the partial quotients, it is natural to ask for the statistical behaviour of this map – especially as we are interested in typical and a-typical behaviour of the sequence of partial quotients.

The ergodic theorem requires the map T to be measure preserving, and it is more or less self-evident that the Gauss map does not preserve the Lebesgue measure. However, there is a measure, which is absolutely continuous and with respect to which the Gauss map is ergodic. There are good reasons why this is the correct measure, although it looks slightly mysterious at first sight, but we will not have time to go into these aspects here.

Definition 4.3. Let $\mathcal{B}$ be the Borel σ-algebra in $[0,1)$. The *Gauss measure* is defined to be the function $\mu : \mathcal{B} \to [0,1]$ defined by

$$\mu(A) = \frac{1}{\log 2}\int_A \frac{1}{1+t}dt = \frac{1}{\log 2}\int_0^1 \chi_A(t)\frac{1}{1+t}dt.$$

Theorem 4.4. *The Gauss measure is preserved under the Gauss map, i.e. for any measurable set A, we have $\mu(T^{-1}A) = \mu(A)$.*

Proof. We note that it is sufficient to prove that $\mu(T^{-1}[0,y)) = \mu([0,y))$, as we can build any other set from basic set operations on these sets. If one considers the graph of the Gauss map (try drawing it), it is easy to see that

$$T^{-1}([0,y)) = \{x \in [0,1) : 0 \le T(x) < y\} = \bigcup_{k=1}^{\infty}\left[\frac{1}{k+y}, \frac{1}{k}\right). \tag{4.3}$$

Thus,

$$\begin{aligned}\mu\left(T^{-1}[0,y)\right) &= \sum_{k=1}^{\infty}\mu\left(\left[\frac{1}{k+y}, \frac{1}{k}\right)\right) = \sum_{k=1}^{\infty}\frac{1}{\log 2}\int_{1/(k+y)}^{1/k}\frac{1}{1+x}dx \\ &= \sum_{k=1}^{\infty}\frac{1}{\log 2}\left[\log\left(1+\frac{1}{k}\right) - \log\left(1+\frac{1}{k+y}\right)\right] \\ &= \sum_{k=1}^{\infty}\frac{1}{\log 2}\log\left(\frac{k+1}{k}\cdot\frac{k+y}{k+y+1}\right).\end{aligned}$$

This is completely incomprehensible, so we try to get to the same incomprehensible expression from the other side. Cunningly, we make an appropriate partition and get

$$\begin{aligned}\mu([0,y)) &= \frac{1}{\log 2}\int_0^y \frac{1}{1+x}dx = \sum_{k=1}^{\infty}\frac{1}{\log 2}\int_{y/(k+1)}^{y/k}\frac{1}{1+x}dx \\ &= \sum_{k=1}^{\infty}\frac{1}{\log 2}\left[\log\left(1+\frac{y}{k}\right) - \log\left(1+\frac{y}{k+1}\right)\right] \\ &= \sum_{k=1}^{\infty}\frac{1}{\log 2}\log\left(\frac{k+1}{k}\cdot\frac{k+y}{k+y+1}\right).\end{aligned}$$

Luckily, this is the same incomprehensible mess that we have before, so the proof is complete. □

Note that the density of the Gauss measure with respect to the Lebesgue measure is continuous, non-negative and in fact invertible. Hence, the two measures are absolutely continuous with respect to each other, and the property of being null or full with respect to one measure automatically implies the same for the other.

We have turned the study of the typical behaviour of continued fractions into a matter of studying the measure preserving system $([0,1), \mathcal{B}, \mu, T)$, where $\mathcal{B}$ is the Borel σ-algebra, μ is the Gauss measure and T is the Gauss map. We have also seen that the Birkhoff ergodic theorem is a nice way of studying the almost everywhere behaviour of such maps. It would be desirable if the measure preserving system we have obtained turned out to be ergodic. It turns out that this is in fact the case. We will prove this now.

First, let us see what the Gauss map does to a continued fraction.

Proposition 4.5. *Let $x = [a_1, a_2, \dots] \in [0,1)$. Then*

$$Tx = T[a_1, a_2, \dots] = [a_2, a_3, \dots].$$

Proof. We see that

$$T[a_1, a_2, \dots] = T\left(\cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \cdots}}}\right)$$
$$= \left\{a_1 + \cfrac{1}{a_2 + \frac{1}{a_3+\cdots}}\right\} = \cfrac{1}{a_2 + \frac{1}{a_3+\cdots}} = [a_2, a_3, \dots]$$

□

This is of course obvious from the construction. But in the light of the measure theoretic considerations, it does actually contain information. We define some sets to make life easier:

Definition 4.6. Let $a_1, \dots, a_n \in \mathbb{N}$. Define the *fundamental interval* or *fundamental cylinder*

$$I_n(a_1, \dots, a_n) = \{[a_1, \dots, a_n, b_{n+1}, b_{n+2}, \dots] : b_{n+i} \in \mathbb{N} \text{ for all } i \in \mathbb{N}\}.$$

Note that by Proposition 4.5, the n'th iterate under the Gauss map of any fundamental cylinder I_n is in fact $[0,1) \setminus \mathbb{Q}$. This reflects the chaotic (or ergodic) nature of the Gauss map. Also note that the cylinders do not include the rational points. This is of little concern to us, as the rationals form a set of measure zero. It does however mean that we have to be extra careful with our bookkeeping.

In the following, let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{N}$ be fixed. Denote by I_n the fundamental interval $I_n(a_1, \ldots, a_n)$. We make a few preliminary observations.

Lemma 4.7. *$x \in I_n$ if and only if there exists $\theta_n(x) \in (0,1) \setminus \mathbb{Q}$ such that*

$$x = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n + \theta_n(x)}}}.$$

Proof. By definition, $x \in I_n$ if and only if $x = [a_1, \ldots, a_n, b_{n+1}, \ldots]$. Applying the Gauss map n times, we get

$$\theta_n(x) := T^n x = [b_{n+1}, b_{n+2}, \ldots].$$

But this is rational if and only if the sequence of partial quotients b_{n+i} terminates. □

The above lemma defines a function on the irrational points in the unit interval $\theta_n : [0,1) \setminus \mathbb{Q} \to [0,1) \setminus \mathbb{Q}$.

Lemma 4.8. *Let $u, v \in [0,1) \setminus \mathbb{Q}$, $u \leq v$. Then*

$$|I_n \cap T^{-n}[u,v]| = |\theta_n^{-1}(v) - \theta_n^{-1}(u)|.$$

Proof. We see that

$$I_n \cap T^{-n}[u,v] = \{x \in [0,1) \setminus \mathbb{Q} : x = [a_1, \ldots, a_n; \theta]\}$$

where $\theta \in [u,v]$ and $[a_1, \ldots, a_n; \theta]$ denotes the continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n + \theta}}}. \tag{4.4}$$

That is, for any $x \in I_n \cap T^{-n}[u,v]$, $\theta_n(x) \in [u,v]$, so

$$I_n \cap T^{-n}[u,v] \subseteq \theta_n^{-1}[u,v].$$

Furthermore, any value of θ in $[u,v]$ inserted in (4.4) will give rise to an element in $I_n \cap T^{-n}[u,v]$, so the converse inclusion holds. Finally, it is an easy exercise left to the reader to see that θ_n^{-1} is monotonic, so this proves the lemma. □

It would seem a good idea to find a precise expression for $\theta_n^{-1}(x)$. This may be done from the recursive formulae for the convergents of x.

Lemma 4.9.

$$\theta_n^{-1}(x) = \frac{p_n + xp_{n-1}}{q_n + xq_{n-1}}.$$

Proof. We prove this by induction in n. For $n = 1$, using Proposition 4.1

$$\frac{p_1 + xp_0}{q_1 + xq_0} = \frac{a_1 p_0 + p_{-1} + xp_0}{a_1 q_0 + q_{-1} + xq_0} = \frac{0 + 1 + x \cdot 0}{a_1 + 0 + x} = \frac{1}{a_1 + x}$$

so $\theta_1^{-1}(x) = (p_1 + xp_0)/(q_1 + xq_0)$.

Now, we consider $n + 1$. We let $y = 1/(a_{n+1} + x)$. We know that

$$\theta_{n+1}^{-1}(x) = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \frac{1}{a_{n+1}+x}}}} = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n + y}}} = \theta_n^{-1}(y).$$

By induction hypothesis and Proposition 4.1 again,

$$\theta_n^{-1}(y) = \frac{p_n + yp_{n-1}}{q_n + yq_{n-1}} = \frac{p_n + \left(\frac{1}{a_{n+1}+x}\right) p_{n-1}}{q_n + \left(\frac{1}{a_{n+1}+x}\right) q_{n-1}}$$
$$= \frac{a_{n+1}p_n + p_{n-1} + xp_n}{a_{n+1}q_n + q_{n-1} + xq_n} = \frac{p_{n+1} + xp_n}{q_{n+1} + xq_n}.$$

This completes the proof. □

Note that θ_n^{-1} is continuous, so we may extend it to all of $[0, 1]$ and still have Lemma 4.8. We now introduce the so-called Vinogradov notation to make our notation less cumbersome.

Definition 4.10. For two real expressions x and y, we say that $x \ll y$ if there exists a constant $c > 0$ such that $x \leq cy$. If $x \ll y$ and $y \ll x$ we write $x \asymp y$.

Lemma 4.11. *Let $u, v \in [0, 1)$ with $u \leq v$. Then*

$$\frac{|T^{-n}[u, v] \cap I_n|}{|I_n|} \asymp |[u, v]|,$$

where the implied constants in $\asymp$ do not depend on the sequence (a_n) defining the I_n.

Proof. We use Lemma 4.8 to obtain

$$\frac{|T^{-n}[u,v]\cap I_n|}{|I_n|} = \left|\frac{\theta_n^{-1}(v)-\theta_n^{-1}(u)}{\theta_n^{-1}(1)-\theta_n^{-1}(0)}\right| = \left|\frac{\frac{p_n+vp_{n-1}}{q_n+vq_{n-1}} - \frac{p_n+up_{n-1}}{q_n+uq_{n-1}}}{\frac{p_n+p_{n-1}}{q_n+q_{n-1}} - \frac{p_n}{q_n}}\right|$$
$$= (v-u)\left|\frac{q_n(q_n+q_{n-1})}{(q_n+vq_{n-1})(q_n+uq_{n-1})}\right|.$$

The last reduction requires substantial, but completely elementary calculations using Proposition 4.1.

Now, the denominators of the convergents q_n satisfy the inequality $q_{n-1}/q_n < 1$, so it is easy to see by Proposition 4.1 (i) that

$$\left|\frac{q_n(q_n+q_{n-1})}{(q_n+vq_{n-1})(q_n+uq_{n-1})}\right| \asymp 1.$$

As $|[u,v]| = v-u$, the proof is completed. □

Lemma 4.12. *For every $A\in\mathcal{B}$,*

$$\frac{\mu(T^{-n}A\cap I_n)}{\mu(I_n)} \asymp \mu(A).$$

Proof. As the Borel σ-algebra is generated by intervals, by Lemma 4.11 for any $A\in\mathcal{B}$,

$$\frac{|T^{-n}A\cap I_n|}{|I_n|} \asymp |A|. \tag{4.5}$$

Also, since $1/2 < 1/(1+t) \le 1$ for $t\in[0,1)$, we have for any $A\in\mathcal{B}$,

$$\frac{1}{2}|A| = \int_A \frac{1}{2}dt \le \int_A \frac{1}{1+t}dt = \mu(A)$$

and

$$\mu(A) = \int_A \frac{1}{1+t}dt \le \int_A 1dt = |A|.$$

Hence, $\mu(A)\asymp|A|$, so the Lemma follows from (4.5). □

Theorem 4.13. *The Gauss map is ergodic with respect to the Gauss measure.*

Proof. Suppose that $T^{-1}A = A$ and that $\mu(A) > 0$. It suffices to prove that $\mu(A) = 1$. Any Borel set can be generated by the I_n, as these intervals are essentially disjoint with lengths tending to zero. Hence, by generating a set

B by I_n's of the same level (up to an arbitrarily small error), Lemma 4.12 implies that

$$\mu(T^{-n}A \cap B) \asymp \mu(A)\mu(B)$$

for any $B \in \mathcal{B}$. On the other hand, as $T^{-1}A = A$,

$$\mu(T^{-n}A \cap B) = \mu(A \cap B)$$

Letting $B = A^c$, we see that $\mu(A \cap B) = 0$, so that $\mu(B) \asymp 0$. This clearly implies that $\mu(B) = \mu(A^c) = 0$, so that $\mu(A) = 1$. $\square$

We specify the following corollary of the Birkhoff Ergodic Theorem and Theorem 4.13:

Corollary 4.14. *Let f be an integrable function on $[0,1)$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \frac{1}{\log 2} \int_0^1 \frac{f(t)}{1+t} dt$$

for almost every $x \in [0,1)$.

This follows only because Lebesgue measure and Gauss measure are absolutely continuous with respect to each other.

We end this course by proving a theorem due to Lévy.

Theorem 4.15. *For almost every $x \in [0,1)$, the sequence of denominators of convergents (q_n) satisfies that*

$$\lim_{n\to\infty} \frac{1}{n} \log q_n = \frac{\pi^2}{12 \log 2}.$$

Proof. We need to get this into a form, where we can apply Birkhoff's ergodic theorem. Let $q_n(x)$ denote the denominator of the n'th convergent of x and similarly for $p_n(x)$. We first notice that

$$\begin{aligned} \frac{p_n(x)}{q_n(x)} &= \frac{1}{a_1 + [a_2, a_3, \dots, a_n]} \\ &= \frac{1}{a_1 + \frac{p_{n-1}(Tx)}{q_{n-1}(Tx)}} \\ &= \frac{q_{n-1}(Tx)}{p_{n-1}(Tx) + a_1 q_{n-1}(Tx)}. \end{aligned}$$

Since convergents are on lowest terms, this implies that $p_n(x) = q_{n-1}(Tx)$.

Now, $p_1 = q_0 = 1$ for all x, so this implies that

$$\frac{1}{q_n(x)} = \frac{p_n(x)}{q_n(x)} \cdot \frac{p_{n-1}(Tx)}{q_{n-1}(Tx)} \cdots \frac{p_1(T^{n-1}x)}{q_1(T^{n-1}x)}.$$

so that

$$-\frac{1}{n} \log q_n(x) = \frac{1}{n} \sum_{j=1}^{n-1} \log \left(\frac{p_{n-j}(T^j x)}{q_{n-j}(T^j x)} \right).$$

It is reasonable to suspect that each summand is pretty close to $\log(T^j x)$, so we set $h(x) = \log x$ and note that $h \in L^1(\mu)$. Then,

$$-\frac{1}{n} \log q_n(x) = \frac{1}{n} \sum_{j=1}^{n-1} \log(T^j x) - \frac{1}{n} \sum_{j=1}^{n-1} \left[\log(T^j x) - \log \left(\frac{p_{n-j}(T^j x)}{q_{n-j}(T^j x)} \right) \right].$$

The first sum is a plain ergodic average, so by the Birkhoff ergodic theorem,

$$\frac{1}{n} \sum_{j=1}^{n-1} \log(T^j x) \to \frac{1}{\log 2} \int_0^1 \frac{\log x}{1+x} dx = -\frac{\pi^2}{12 \log 2},$$

by Exercise 4.4. We must show that the second sum tends to 0.

From Proposition 4.1,

$$p_k \geq 2^{(k-2)/2}, \quad q_k \geq 2^{(k-1)/2},$$

so using again Proposition 4.1,

$$\left| \frac{x}{p_k/q_k} - 1 \right| = \frac{q_k}{p_k} \left| x - \frac{p_k}{q_k} \right| \leq \frac{1}{p_k q_{k+1}} \leq \frac{1}{2^{k-1}}.$$

Furthermore, if $u \in [\frac{1}{2}, \frac{3}{2}]$, then $|\log u| \leq 2|u-1|$. We apply this to the summands, for which it is valid to obtain

$$\left| \sum_{j=1}^{n-1} \left[\log(T^j x) - \log \left(\frac{p_{n-j}(T^j x)}{q_{n-j}(T^j x)} \right) \right] \right| \leq \sum_{j=1}^{n-1} \left| \log \frac{T^j x}{p_{n-j}(T^j x)/q_{n-j}(T^j x)} \right|$$
$$\leq 2 \sum_{j=1}^{n-2} \left| \frac{T^j x}{p_{n-j}(T^j x)/q_{n-j}(T^j x)} \right| + \left| \log \frac{T^{n-1} x}{p_1(T^{n-1}x)/q_1(T^{n-1}x)} \right|.$$

The first sum satisfies that

$$2 \sum_{j=1}^{n-2} \left| \frac{T^j x}{p_{n-j}(T^j x)/q_{n-j}(T^j x)} \right| \leq 2 \sum_{j=1}^{n-2} \frac{1}{2^{n-j-1}} \leq 2,$$

for all $n \geq 3$. For the second term, observe that

$$\left|\log \frac{T^{n-1}x}{p_1(T^{n-1}x)/q_1(T^{n-1}x)}\right| = \left|\log\left((T^{n-1}x)a_1(T^{n-1}x)\right)\right|,$$

since $p_1 = 1$ and q_1 is the first partial quotient. But one easily sees that for any num(ber $x \in [0,1)$, $xa_1(x) \in [\frac{1}{2}, 1]$, so

$$\left|\log\left((T^{n-1}x)a_1(T^{n-1}x)\right)\right| \leq \log 2.$$

It follows that

$$\frac{1}{n}\sum_{j=1}^{n-1}\left[\log(T^j x) - \log\left(\frac{p_{n-j}(T^j x)}{q_{n-j}(T^j x)}\right)\right] \leq \frac{2+\log 2}{n} \to 0.$$

□

A final corollary gives us an expected average speed of convergence for the simple continued fraction.

Corollary 4.16. *For almost all $x \in [0,1]$,*

$$\lim_{n\to\infty} \frac{1}{n}\log\left|x - \frac{p_n}{q_n}\right| \to -\frac{\pi^2}{6\log 2}.$$

Proof. This follows immediately from Lévy's Theorem and Proposition 4.1, since

$$\log q_n + \log q_{n+1} \leq -\log\left|x - \frac{p_n}{q_n}\right| \leq \log q_n + \log q_{n+2}.$$

□

Exercises

Exercises to all sections

4.1 Use the Birkhoff ergodic theorem to show that the density of the partial quotient j in almost all $x \in [0,1)$ is equal to

$$\frac{2\log(1+j) - \log j - \log(2+j)}{\log 2}.$$

4.2 Show that the geometric mean of the partial quotients for almost all $x \in [0,1)$ converges in the sense that

$$\lim_{n\to\infty}(a_1 a_2 \cdots a_n)^{1/n} = \prod_{a=1}^{\infty}\left(\frac{(a+1)^2}{a(a+2)}\right)^{\log a/\log 2}.$$

4.3 Show that the arithmetic mean of the partial quotients of almost all numbers $x \in [0, 1)$ diverges, i.e. that

$$\lim_{n\to\infty} \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = \infty.$$

(*Hint:* Be careful! The first function which comes to mind is not in $L^1(\mu)$!)

4.4 This exercise is really just in integration, but fun nonetheless. Show that

$$\int_0^1 \frac{\log x}{1+x} dx = -\frac{\pi^2}{12}.$$

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