

ALGEBRAIC AND HOMOLOGICAL PROPERTIES OF POWERS AND SYMBOLIC POWERS OF IDEALS

JÜRGEN HERZOG

Lahore, February 21–28, 2009

1. LECTURE: ON THE REGULARITY OF POWERS OF AN IDEAL

In this lecture we study the asymptotic behaviour of Castelnuovo–Mumford regularity of powers of a graded ideal I and give a proof of the theorem, proved by Cutkosky, Herzog, Trung [3] and Kodiyalam [5], that the regularity of I^k is a linear function of k for large k . First we recall some basic facts from commutative homological algebra. We consider graded free resolutions, regularity and projective dimension of a graded ideal and the Rees algebra associated to an ideal.

The study of powers of an ideal was initiated by a result of Bertram, Ein and Lazarsfeld [1] who proved the following vanishing theorem: let X be a smooth projective variety, and let d_X denote the minimum of the degrees d such that X is a scheme-theoretic intersection of hypersurfaces of degree at most d . Then there is a number e such that

$$H^i(\mathbb{P}^s, \mathcal{I}_X^n(a)) = 0 \quad \text{for all } a \geq nd_X + e, \quad i \geq 1.$$

The proof uses the Kodaira vanishing theorem. Later Swanson [6] showed that for any graded ideal $I \subset K[x_1, \dots, x_n]$ the regularity of its powers are bounded by a linear function. In case that $\dim S/I$ one even has $\text{reg}(I^k) \leq k \text{reg}(I)$, as was shown by Geramita, Gimigliano and Pitteloud [4] and Chandler [2].

REFERENCES

- [1] A. Bertram, L. Ein, and R. Lazarsfeld, Vanishing theorems, a theorem of Severi, and the equations defining projective varieties. *J. Amer. Math. Soc.* **4**, 587–602, (1991).
- [2] K. A. Chandler, Regularity of the powers of an ideal. *Commun. Algebra* **25**, 3773–3776, (1997)
- [3] D. Cutkosky - J. Herzog - N.V. Trung, Asymptotic behaviour of the Castelnuovo-Mumford regularity. *Compositio Math.* **118**, 243–261, (1999)
- [4] A. V. Geramita, A. Gimigliano and Y. Pitteloud, Graded Betti numbers of some embedded rational n -folds. *Math. Ann.* **301**, 363–380, (1995)
- [5] V. Kodiyalam, Asymptotic behaviour of Castelnuovo-Mumford regularity. *Proceedings of the AMS* **128**, 407–411, (1999)
- [6] I. Swanson, Powers of ideals, primary decompositions, Artin–Rees lemma and regularity. *Math. Ann.* **307**, 299–313, (1997)

2. LECTURE: POWERS OF IDEALS WITH LINEAR RESOLUTION AND LINEAR QUOTIENTS

If an ideal I has a linear resolution, then this does not imply that all powers I^k of I have a linear resolution. The Stanley–Reisner ideal of the projective plane is such an example, as was first observed by Terai. There are examples given by Conca [4] which demonstrate that the powers of an ideal may all have a linear resolution up to given power k , but then fail to have a linear resolution for the $(k + 1)$ th power. On the other hand it will be shown that if the defining ideal J of the Rees algebra $R(I)$ of I satisfies a certain Gröbner basis condition, then all powers do indeed have a linear resolution. A somewhat stronger condition on the Gröbner basis makes even sure that all powers of I have linear quotients. Finally we discuss ideals whose powers are componentwise linear and present some open conjectures.

REFERENCES

- [1] A. Conca, J. Herzog, Castelnuovo-Mumford regularity of products of ideals. *Collect. Math.* **54** 137–152, (2003)
- [2] J. Herzog, T. Hibi, X. Zheng, Monomial ideals whose powers have a linear resolution. *Math. Scand.* **95**, 23–32, (2004)
- [3] T. Römer, Homological properties of bigraded algebras. III. *J. Math.* **45**, 1361–1376, (2001)
- [4] A. Conca, Regularity jumps for powers of ideals. *Proceedings Lisbon Conference on Commutative Algebra, Lisbon-Portugal, 2003.*
- [5] D. Eisenbud, C. Huneke, B. Ulrich, The regularity of Tor and graded Betti numbers. *Amer. J. Math.* **128**, 573–605, (2006)

3. LECTURE: ON THE THE GROWTH OF THE BETTI NUMBERS AND THE DEPTH OF POWERS OF AN IDEAL

Let S be either a Noetherian local ring or a standard graded K -algebra, and let $I \subset S$ be a (graded) ideal. A classical result by Burch [4] says that $\min_k \text{depth } S/I^k \leq d - \ell(I)$, where $\ell(I)$ the analytic spread of I . By a theorem of Brodmann [3], $\text{depth } S/I^k$ is constant for $k \gg 0$. We call this constant value the limit depth of I , and denote it by $\lim_{k \rightarrow \infty} \text{depth } S/I^k$. Brodmann improved the Burch inequality by showing that $\lim_{k \rightarrow \infty} \text{depth } S/I^k \leq d - \ell(I)$. Eisenbud and Huneke [5] showed that equality holds, if the associated graded ring $\text{gr}_I(S)$ is Cohen–Macaulay. This is for example the case if S and $\mathcal{R}(I)$ are Cohen–Macaulay, see Huneke [6]. We will present a proof of these facts. Then we discuss the initial behaviour of the function $k \mapsto \text{depth } I^k$. If all powers have a linear resolution, then $\text{depth } S/I^k$ is a nonincreasing function. Which nonincreasing functions are possible? Denote by Δf the difference function of f . It will be shown that for given nonincreasing function $f: \mathbb{N} \rightarrow \mathbb{N}$ with $f(0) = 2 \lim_{k \rightarrow \infty} f(k) + 1$ for which Δf is nonincreasing, there exists a monomial ideal $I \subset S$ such that $\text{depth } S/I^k = f(k)$ for all $k \geq 1$. On the other hand, it can be shown that given any bounded nondecreasing function $f: \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$, there exists a monomial ideal I such that $\text{depth } S/I^k = f(k)$ for all k . It remains the open question whether any eventually constant function $f: \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$ is the depth function of some ideal.

REFERENCES

- [1] L. Burch, Codimension and analytic spread. *Proc. Cambridge Philos. Soc.* **72**, 369–373, (1972)

- [2] M. Brodmann, Asymptotic stability of $\text{Ass}(M/I^n M)$. Proc. Amer. Math. Soc. **74**, 16–18, (1979)
- [3] M. Brodmann, The asymptotic nature of the analytic spread. Math. Proc. Cambridge Philos. Soc. **86**, 35–39, (1979)
- [4] L. Burch, Codimension and analytic spread. Proc. Cambridge Philos. Soc. **72**, 369–373, (1972)
- [5] D. Eisenbud and C. Huneke, Cohen-Macaulay Rees algebras and their specialization. J. Algebra **81**, 202–224, (1983)
- [6] C. Huneke, On the associated graded ring of an ideal. Illinois J. Math. **26**, 121–137, (1982)
- [7] J. Herzog - T. Hibi, The depth of powers of an ideal. J. Algebra **291**, 534–550, (2005)

4. LECTURE: SYMBOLIC POWERS

Let K be an algebraically closed field, $Y \subset \mathbb{A}^n$ an affine variety, $P = Z(Y)$ the vanishing ideal of Y and $a = (a_1, \dots, a_n) \in \mathbb{A}^n$ a point. We say that $f \in S = K[x_1, \dots, x_n]$ vanishes at a of order $\geq k$, if $f \in \mathfrak{m}_a^k$, where $\mathfrak{m}_a = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$, and set $P^{(k)} = \{f \in S : f \text{ vanishes of order } \geq k \text{ at every point of } Y\}$. On the other hand, one defines the k th symbolic power $P^{(k)}$ of P as follows: $P^{(k)} = \{f \in S : fg \in P^k \text{ for some } g \notin P\}$. It is easily seen that $P^{(k)} = \text{Ker}(S \rightarrow S_P/P^k S_P)$. We have $P^{(1)} = P$ and $P^k \subseteq P^{(k)}$. In this lecture we first prove the theorem of Zariski-Nagata which says that $P^{(k)} = P^k$. A generalization of this theorem was given by Eisenbud and Hochster [3]. What can be said about the regularity of the symbolic powers? In general the symbolic Rees algebra is not finitely generated, see [1], [2], [6] and [7]. We introduce generalized symbolic powers and show that for monomial ideals these powers have a finitely generated symbolic Rees algebra. Consequently the regularity of these powers is linearly bounded. Other cases for which there exists linear bounds are given in the paper [5].

REFERENCES

- [1] D. Cutkosky, Symbolic algebras of monomial primes, J. Reine Angew. Math. 416 (1991) 71–89.
- [2] S. Goto, K. Nishida, K. Watanabe, Non-Cohen-Macaulay symbolic Rees algebras for space monomial curves and counterexamples to Cowsik’s question. Proc. Amer. Math. Soc. **120**, 383–392, (1994)
- [3] D. Eisenbud, M. Hochster, A Nullstellensatz with nilpotents. J. Algebra **58**, 157–161, (1979)
- [4] J. Herzog, T. Hibi, N.V. Trung, Symbolic powers of monomial ideals and vertex cover algebras. Adv. Math. **210**, 304–322, (2007)
- [5] Herzog, Le T. Hoa, N.V. Trung, Asymptotic linear bounds for the Castelnuovo-Mumford regularity. Trans. Amer. Math. Soc. **354**, 1793–1809, (2002)
- [6] C. Huneke, Hilbert functions and symbolic powers, Michigan Math. J. **34**, 293–318, (1987)
- [7] P. Roberts, A prime ideal in a polynomial ring whose symbolic blow-up is not Noetherian, Proc. Amer. Math. Soc. **94**, 589–592, (1985)