STANLEY DECOMPOSITIONS AND STANLEY'S CONJECTURE

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1. LECTURE: STANLEY DECOMPOSITIONS AND PRIME FILTRATIONS (M. VLADOIU)

In this lecture we study the conjecture of Stanley [4]. Let K be a field, $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables, and M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element in M and Z a subset of $\{x_1, \dots, x_n\}$. We denote by $uK[Z]$ the K-subspace of M generated by all elements uv where v is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded K-subspace $uK[Z] \subset M$ is called a *Stanley space of dimension* $|Z|$, if $uK[Z]$ is a free $K[Z]$-module.

A *Stanley decomposition* of M is a presentation of the $\mathbb{Z}^n$-graded K-vector space M as a finite direct sum of Stanley spaces

$$\mathscr{D} : M = \bigoplus_{i=1}^{m} u_i K[Z_i]$$

in the category of $\mathbb{Z}^n$-graded K-vector spaces. The number $\operatorname{sdepth} \mathscr{D} = \min\{|Z_i| : \ i = 1, \dots, m\}$ is called the *Stanley depth of* $\mathscr{D}$. The *Stanley depth* of M is defined to be

$$\operatorname{sdepth} M = \max\{\operatorname{sdepth} \mathscr{D} : \ \mathscr{D} \text{ is a Stanley decomposition of } M\}.$$

Stanley conjectured [4] that $\operatorname{depth} M \leq \operatorname{sdepth} M$ for all $\mathbb{Z}^n$-graded S-modules M.

We discuss in the present lecture the relationship between Stanley decompositions and prime filtrations (see also [1],[2], [4]). We will also recall some basic upper and lower bounds for the Stanley depth of a finitely generated $\mathbb{Z}^n$-graded S-module (see [2]).

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2. LECTURE: STANLEY DECOMPOSITIONS AND PARTITIONS (M. VLADOIU)

The computation of the Stanley depth of an arbitrary finitely generated $\mathbb{Z}^n$-graded S-module is a very hard combinatorial problem, since not all Stanley decompositions arise from prime filtrations ([2]). In general it is not known an algorithm for the computation of the Stanley depth. However, in the particular case when the $\mathbb{Z}^n$-graded S-module is of the form I/J, where $J \subset I$ are monomial ideals of S such an algorithm was done by Herzog & al.in [2]. We present this algorithm, which was later implemented in CoCoA by Rinaldo, see [3]. Even with this algorithm the computation of the Stanley depth of a particular class of monomial ideals is not easy. For example, in the case of the maximal monomial ideal of S it was conjectured in [2] that the Stanley depth is $\lceil n/2 \rceil$ and proved later purely combinatorial by Biro & al. in [1]. For the class of complete intersection monomial ideals the Stanley depth is also known, see [4].

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3. LECTURE: STANLEY DEPTH OF MULTIGRADED MODULES (D. POPESCU)

Since any prime filtration of a $\mathbb{Z}^n$-graded finitely generated S-module M defines a Stanley decomposition we may define the fdepth of M as the maximum Stanley depth of those Stanley decompositions induced by prime filtrations. First we recall some basic inequalities between fdepth, depth and Stanley depth, see for example [2], [3], [5]. The aim of this lecture is to show that Stanley's conjecture holds for multigraded modules of type $M = J/I$, where $I \subset J$ are monomial ideals of S such that I is reduced and $\dim M \leq 2$.

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4. Lecture: Stanley's Conjecture For Monomial Square-free Ideals (D. Popescu)

In this lecture we give a proof of the Stanley's Conjecture in the particular case of monomial square free ideals of $S = K[X_1, \ldots, X_n]$ for $n \leq 5$. The lecture is mainly based on [2], but we also rely on some results of A. Rauf and Herzog & al.

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