STANLEY DECOMPOSITIONS AND STANLEY'S CONJECTURE

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1. Lecture: Stanley decompositions and prime filtrations

Let K be a field, $S = K[x_1,\dots,x_n]$ be the polynomial ring in n variables, and M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element in M and Z a subset of $\{x_1,\dots,x_n\}$. We denote by $uK[Z]$ the K-subspace of M generated by all elements uv where v is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded K-subspace $uK[Z] \subset M$ is called a *Stanley space of dimension* $|Z|$, if $uK[Z]$ is a free $K[Z]$-module.

A *Stanley decomposition* of M is a presentation of the $\mathbb{Z}^n$-graded K-vector space M as a finite direct sum of Stanley spaces

$$\mathscr{D} : M = \bigoplus_{i=1}^{m} u_i K[Z_i]$$

in the category of $\mathbb{Z}^n$-graded K-vector spaces. In other words, each of the summands is a $\mathbb{Z}^n$-graded K-subspace of M and the decomposition is compatible with the $\mathbb{Z}^n$-grading, i.e. for each $a \in \mathbb{Z}^n$ we have $M_a = \bigoplus_{i=1}^m (u_i K[Z_i])_a$. The number $\operatorname{sdepth}\mathscr{D} = \min\{|Z_i| : i = 1,\dots,m\}$ is called the *Stanley depth of* $\mathscr{D}$. The *Stanley depth* of M is defined to be

$$\operatorname{sdepth} M = \max\{\operatorname{sdepth}\mathscr{D} : \mathscr{D} \text{ is a Stanley decomposition of } M\}.$$

It is conjectured by Stanley [16] that $\operatorname{depth} M \le \operatorname{sdepth} M$ for all $\mathbb{Z}^n$-graded S-modules M. The conjecture is widely open (see however [1], [2], [7], [6], [11] and [14]).

In this first lecture we present some general facts about arbitrary $\mathbb{Z}^n$-graded S-modules M. More precisely, we first recall the existence of a Stanley decomposition for such a module, which is induced by any prime filtration of the module. For the later applications we also give a simple characterization of Stanley decompositions induced by a prime filtration. Later on, we give some general upper and lower bounds for the depth and sdepth of a module, as well as their connection with another invariant: the fdepth of a module.

How can we compute a Stanley decomposition of an arbitrary $\mathbb{Z}^n$-graded S-modules M? Let us start with an example. Figure 1 displays a Stanley decomposition of S/I and of I for the monomial ideal $I = (x_1x_2^3, x_1^2x_2^2, x_1^3x_2)$. The gray area represents the K-vector space spanned by the monomials in I. The hatched area, the fat lines and the isolated fat dots represent Stanley spaces of dimension 2,1, and 0, respectively. According to Figure 1 we have the following Stanley decompositions

$$I = x_1x_2^3K[x_1,x_2] \oplus x_1^2x_2^2K[x_1] \oplus x_1^3x_2K[x_1],$$

and

$$S/I = K[x_2] \oplus x_1K[x_1] \oplus x_1x_2K \oplus x_1x_2^2K \oplus x_1^2x_2K.$$

Here we identify S/I with the K-subspace of S generated by all monomials $u \in S \setminus I$.

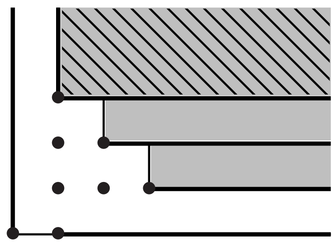

FIGURE 1

In general we have the following

Lemma 1.1. *Any finitely generated $\mathbb{Z}^n$-graded S-module M admits a Stanley decomposition.*

The proof is based on the fact that any prime filtration of M yields a Stanley decomposition. We call a chain of $\mathbb{Z}^n$-graded submodules

$$\mathscr{F}: 0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

a *prime filtration* of M if $M_i/M_{i-1} \cong (S/P_i)(-a_i)$ where $a_i \in \mathbb{Z}^n$ and where each P_i is a monomial prime ideal. We call the set of prime ideals $\{P_1,\ldots,P_m\}$ the *support* of $\mathscr{F}$ and denote it by $\operatorname{supp}(\mathscr{F})$. It is a well-known fact in commutative algebra that at least one such prime filtration always exists. Each prime filtration $\mathscr{F}$ of M gives rise to a Stanley decomposition $\mathscr{D}(\mathscr{F})$ as follows:

Since $M_i/M_{i-1} \cong S/P_i(-a_i)$, there exists a homogeneous element $u_i \in M_i$ of degree a_i, whose residue class modulo M_{i-1} generates M_i/M_{i-1} and such that $u_iK[Z_i] \cong M_i/M_{i-1}$, where $Z_i = \{x_j : \; x_j \notin P_i\}$ and where $u_iK[Z_i]$ is a free $K[Z_i]$-module. The filtration $\mathscr{F}$ provides a decomposition $M = \bigoplus_{i=1}^m M_i/M_{i-1}$ as direct sum of K-vector spaces. Since each of the factors M_i/M_{i-1} is a Stanley space $u_iK[Z_i]$ we obtain the decomposition $\mathscr{D}(\mathscr{F}) = \bigoplus_{i=1}^m u_iK[Z_i]$, as desired. We say that $\mathscr{D}(\mathscr{F})$ is the Stanley decomposition induced by the prime filtration $\mathscr{F}$. Not all Stanley decompositions of M are induced by prime filtrations (see [9],[8]), therefore $\operatorname{sdepth}(M)$ can not be computed using only the Stanley decompositions arising from prime filtrations. However, we can describe those Stanley decompositions arising from prime filtrations as follows

Proposition 1.2. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module and $\mathscr{D} : M = \bigoplus_{i=1}^m u_i K[Z_i]$ a Stanley decomposition of M. Then the following conditions are equivalent:*
(a) *$\mathscr{D}$ is induced by a prime filtration.*
(b) *After a suitable relabeling of the summands in $\mathscr{D}$ we have $M_j = \bigoplus_{i=1}^j u_i K[Z_i]$ is a $\mathbb{Z}^n$-graded submodule of M for $j = 1, \ldots, m$.*

Proof. (a) $\Rightarrow$ (b) follows immediately from the construction of a Stanley decomposition which is induced by a prime filtration.

(b) $\Rightarrow$ (a): We claim that $\mathscr{F} : 0 \subset M_1 \subset M_2 \subset \ldots \subset M_m = M$ is a prime filtration of M. First notice that for each j, the module M_j/M_{j-1} is a cyclic module generated by the residue class $\bar{u_j} = u_j + M_{j-1}$. Indeed, each element $u \in M_j$ can be written as $u = \sum_{k=1}^j u_k f_k$ with $f_k \in K[Z_k]$ for $k = 1, \ldots, j$. Therefore $\bar{u} = \bar{u_j} f_j$.

Next we claim that the annihilator of $\bar{u_j}$ is equal to the monomial prime ideal P generated by the variables $x_k \notin Z_j$. In fact, if $x_k \notin Z_j$, then $\deg x_k u_j \neq \deg u_j v$ for all monomials $v \in K[Z_j]$. Therefore, since $M_j = \bigoplus_{i=1}^j u_i K[Z_i]$ is a decomposition of $\mathbb{Z}^n$-graded K-vector spaces, it follows that $x_k u_j \in M_{j-1}$. This implies that $x_k \bar{u_j} = 0$ and shows that P is contained in the annihilator of $\bar{u_j}$. On the other hand, if v is a monomial in $S \setminus P$, then $v \in K[Z_j]$ and so $u_j v$ is a nonzero element in $u_j K[Z_j]$. This implies that v does not belong to the annihilator of $\bar{u_j}$ and shows that P is precisely the annihilator of $\bar{u_j}$. From all this we conclude that $\mathscr{D}$ is induced by $\mathscr{F}$. □

Based on the Stanley decompositions arising from prime filtrations we can obtain a lower bound for the sdepth. More precisely, given a filtration $\mathscr{F}$ of a module M one can define $\operatorname{fdepth} \mathscr{F} = \min\{\dim S/P : P \in \operatorname{supp} \mathscr{F}\}$ and

$$\operatorname{fdepth} M = \max\{\operatorname{fdepth} \mathscr{F} : \mathscr{F} \text{ is a prime filtration of } M\}.$$

It follows from the discussion before Proposition 1.2 that $\operatorname{fdepth} \mathscr{F} = \operatorname{sdepth} \mathscr{D}(\mathscr{F})$ and consequently

$$\operatorname{fdepth} M \leq \operatorname{sdepth} M.$$

Since Stanley's conjecture states that the following inequality $\operatorname{depth} M \leq \operatorname{sdepth} M$ holds true for any arbitrary $\mathbb{Z}^n$-graded S-module M, we would be interested in obtaining some inequalities involving depth and sdepth. We have the following result

Proposition 1.3. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Then*

$$\operatorname{fdepth} M \leq \operatorname{depth} M \leq \min\{\dim S/P : P \in \operatorname{Ass}(M)\},$$

and $\operatorname{fdepth} M \leq \operatorname{sdepth} M$. *If, in addition* $\dim_K M_a \leq 1$ *for all $a \in \mathbb{Z}^n$ then we also have that*

$$\operatorname{sdepth} M \leq \min\{\dim S/P : P \in \operatorname{Ass}(M)\}.$$

Proof. The bounds for the $\operatorname{depth} M$ are well-known. We will give an argument for the upper bound of $\operatorname{sdepth} M$, due to Soleyman-Jahan [15]. Let $\mathscr{D} : M = \bigoplus_{i=1}^t u_i K[Z_i]$ be a Stanley decomposition of M such that $\operatorname{sdepth} M = \operatorname{sdepth} \mathscr{D}$ and $Q \in \operatorname{Ass}(M)$ such that $\dim S/Q = \min\{\dim S/P : P \in \operatorname{Ass}(M)\}$. Since $Q \in \operatorname{Ass}(M)$ there exists a nonzero homogeneous element $m \in M$ such that $Q = \operatorname{Ann}(m)$. Because of our hypothesis: $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$, it follows that there exists a unique $l \in [t]$ such that $m \in u_l K[Z_l]$. Hence $m = u_l f$ for some monomial $f \in K[Z_l]$. It is enough to show that $Z_l \cap Q = \emptyset$. Suppose that $Z_l \cap Q \neq \emptyset$ and choose $x_i \in Z_l \cap Q$. Then $0 = x_i m = u_l(x_i f) \in u_l K[Z_l]$, a contradiction since $u_l K[Z_l]$ is a free $K[Z_l]$-module. Hence $Z_l \cap Q = \emptyset$ and consequently $|Z_l| \leq \dim S/Q$. Therefore we have

$$\operatorname{sdepth} M = \operatorname{sdepth} \mathscr{D} = \min\{\|Z_i|: \ i \in [t]\} \leq |Z_l| \leq \dim S/Q = \min\{\dim S/P: \ P \in \operatorname{Ass}(M)\}.$$

□

It is clear that whenever $\operatorname{depth} M = \operatorname{fdepth} M$ then Stanley's conjecture holds for M. This situation happens of course if $\operatorname{fdepth} M = \min\{\dim S/P: \ P \in \operatorname{Ass}(M)\}$. This is the case if M admits a prime filtration $\mathscr{F}$ with $\operatorname{supp}(\mathscr{F}) = \operatorname{Ass}(M)$ in which case M is said to be *almost clean*. In the particular case when $M = S/I$, where I is a monomial ideal of S, since $\dim_K(S/I)_a \leq 1$ for all $a \in \mathbb{Z}^n$ we notice by Proposition 1.3 that if S/I is almost clean then $\operatorname{fdepth} S/I = \operatorname{depth} S/I = \operatorname{sdepth} S/I$. For example, when I is of the following type: perfect monomial ideal of codimension 2 or Gorenstein monomial ideal of codimension 3 or ideal of Borel type or $n \leq 3$ it was proved (see [7], [6], [14]) that S/I is almost clean. Consequently, if S/I is almost clean we can actually compute $\operatorname{sdepth} S/I$, which is equal to $\operatorname{depth} S/I$. However, in general we may have strict inequality as the following example shows.

Let $M = \mathfrak{m} = (x_1, x_2, x_3)$ be the maximal ideal of $S = K[x_1, x_2, x_3]$. Then we claim that $\operatorname{depth} \mathfrak{m} = 1 < \operatorname{sdepth} \mathfrak{m} = 2$. The only question is why the Stanley depth of $\mathfrak{m}$ is equal to 2. To see this, we first observe that for a monomial ideal $I \subset K[x_1, \ldots, x_n]$ we have $\operatorname{sdepth} I = n$, if and only if I is a principal ideal. Indeed, if $I = (u)$, then $I = uK[x_1, \ldots, x_n]$ is a Stanley decomposition. On the other hand, if I is not principal at least two Stanley spaces are needed to cover I. Obviously any two Stanley spaces of dimension n intersect, so that one of the summands in the Stanley decomposition must have dimension smaller than n.

Thus we have $\operatorname{sdepth} \mathfrak{m} \leq 2$. Since the following is a Stanley decomposition of $\mathfrak{m}$

$$(1) \qquad (x_1, x_2, x_3) = x_1 x_2 x_3 K[x_1, x_2, x_3] \oplus x_1 K[x_1, x_2] \oplus x_2 K[x_2, x_3] \oplus x_3 K[x_1, x_3].$$

of sdepth 2 it follows that $\operatorname{sdepth} \mathfrak{m} = 2$.

2. Lecture: Stanley decompositions and partitions

In this lecture we present an algorithm (see [8]) for computing in a finite number of steps the $\operatorname{sdepth} M$ in the particular case when M is of the form I/J, where $J \subset I$ are monomial ideals of S. Afterwards we will present some applications of the algorithm following from Theorem 2.1. It is worthwhile to mention that the algorithm was implemented in CoCoA by Rinaldo ([12]).

First, we define a natural partial order on $\mathbb{N}^n$ as follows: $a \leq b$ if and only if $a(i) \leq b(i)$ for $i = 1, \ldots, n$. Note that $x^a | x^b$ if and only if $a \leq b$. Here, for any $c \in \mathbb{N}^n$ we denote as usual by x^c the monomial $x_1^{c(1)} x_2^{c(2)} \cdots x_n^{c(n)}$. Observe that $\mathbb{N}^n$ with the partial order introduced is a distributive lattice with meet $a \wedge b$ and join $a \vee b$ defined as follows: $(a \wedge b)(i) = \min\{a(i), b(i)\}$ and $(a \vee b)(i) = \max\{a(i), b(i)\}$. We also denote by ε_j the jth canonical unit vector in $\mathbb{Z}^n$.

Suppose I is generated by the monomials $x^{a_1}, \ldots, x^{a_r}$ and J by the monomials $x^{b_1}, \ldots, x^{b_s}$. We choose $g \in \mathbb{N}^n$ such that $a_i \leq g$ and $b_j \leq g$ for all i and j, and let $P_{I/J}^g$ be the set of all

$c \in \mathbb{N}^n$ with $c \leq g$ and such that $a_i \leq c$ for some i and $c \not\geq b_j$ for all j. The set $P^g_{I/J}$ viewed as a subposet of $\mathbb{N}^n$ is a finite poset. We call it the *characteristic poset* of I/J with respect to g. There is a natural choice for g, namely the join of all the a_i and b_j. For this g, the poset $P^g_{I/J}$ has the least number of elements, and we denote it simply by $P_{I/J}$. Note that if Δ is a simplicial complex on the vertex set $[n]$, then P_{S/I_Δ} is just the face poset of Δ.

Figure 2 shows the characteristic poset for the maximal ideal $\mathfrak{m} = (x_1, x_2, x_3) \subset K[x_1, x_2, x_3]$. The elements of this poset correspond to the squarefree monomials x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3 and $x_1x_2x_3$. Thus the corresponding labels in Figure 2 should be $(1,0,0), (0,1,0), \ldots, (1,1,1)$ and for short we write $100, 010, \ldots, 111$.

Given any poset P and $a, b \in P$ we set $[a,b] = \{c \in P\colon\ a \leq c \leq b\}$ and call $[a,b]$ an *interval*. Of course, $[a,b] \neq \emptyset$ if and only if $a \leq b$. Suppose P is a finite poset. A *partition* of P is a disjoint union

$$\mathscr{P}\colon\ P = \bigcup_{i=1}^{r} [a_i, b_i]$$

of intervals.

Figure 2 also displays a partition of the characteristic poset for the maximal ideal. The framed regions in Figure 2 indicate that $P_{\mathfrak{m}} = [100, 110] \cup [010, 011] \cup [001, 101] \cup [111, 111]$.

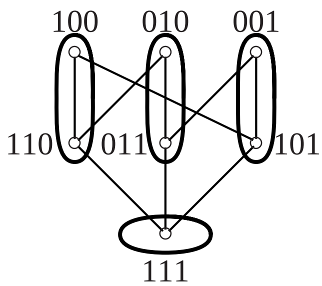

FIGURE 2

We will show that each partition of $P^g_{I/J}$ gives rise to a Stanley decomposition of I/J.

In order to describe the Stanley decomposition of I/J coming from a partition of $P^g_{I/J}$ we shall need the following notation: for each $b \in P^g_{I/J}$, we set $Z_b = \{x_j\colon\ b(j) = g(j)\}$. We then have

Theorem 2.1. *(*[8]*)* (a) *Let* $\mathscr{P}\colon P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ *be a partition of* $P^g_{I/J}$. *Then*

$$\mathscr{D}(\mathscr{P})\colon I/J = \bigoplus_{i=1}^{r} \Big(\bigoplus_{c} x^c K[Z_{d_i}]\Big) \tag{2}$$

is a Stanley decomposition of I/J, *where the inner direct sum is taken over all* $c \in [c_i, d_i]$ *for which* $c(j) = c_i(j)$ *for all* j *with* $x_j \in Z_{d_i}$. *Moreover,* $\operatorname{sdepth} \mathscr{D}(\mathscr{P}) = \min\{|Z_{d_i}|\colon\ i = 1, \ldots, r\}$.

(b) *Let* $\mathscr{D}$ *be a Stanley decomposition of* I/J. *Then there exists a partition* $\mathscr{P}$ *of* $P^g_{I/J}$ *such that*

$$\operatorname{sdepth} \mathscr{D}(\mathscr{P}) \geq \operatorname{sdepth} \mathscr{D}.$$

In particular, $\operatorname{sdepth} I/J$ *can be computed as the maximum of the numbers* $\operatorname{sdepth} \mathscr{D}(\mathscr{P})$*, where* $\mathscr{P}$ *runs over the (finitely many) partitions of* $P_{I/J}^g$.

Proof. (a) We first show that the sum of the K-vector spaces in (2) is equal to the K-vector space spanned by all monomials $u \in I \setminus J$ (which of course is isomorphic to the K-vector space I/J).

Let $u = x^e$ be a monomial in $I \setminus J$ and let $c' = e \wedge g$. Then, $c' \in P_{I/J}^g$ and consequently, there exists $i \in \{1, \ldots, r\}$ such that $c' \in [c_i, d_i]$. Let c be the vector with

$$c(j) = \begin{cases} c_i(j), & \text{if } d_i(j) = g(j), \\ c'(j), & \text{otherwise.} \end{cases}$$

It follows from the definition of c that $x^c K[Z_{d_i}]$ is one of the Stanley spaces appearing in (2). We claim that $u \in x^c K[Z_{d_i}]$, equivalently, that $x^{e-c} \in K[Z_{d_i}]$. Indeed, if $x_j \in Z_{d_i}$, then $d_i(j) = g(j)$, and hence $e(j) \geq c'(j) \geq c_i(j) = c(j)$. On the other hand, if $x_j \notin Z_{d_i}$, then $g(j) > d_i(j) \geq c'(j) = c(j)$. Since $c'(j) = \min\{e(j), g(j)\}$, it therefore follows that $e(j) = c(j)$, as desired.

In order to prove that the sum (2) is direct, it suffices to show that any two different Stanley spaces in (2) have no monomial in common. Suppose to the contrary that $x^b \in x^p K[Z_{d_i}] \cap x^q K[Z_{d_j}]$ and that $x^p K[Z_{d_i}] \neq x^q K[Z_{d_j}]$ are both summands in (2). Since each of the inner sums in (2) is direct, we have that $i \neq j$.

We claim that $x^b \in x^p K[Z_{d_i}]$ yields $b \wedge g \in [c_i, d_i]$. Indeed, since $c_i \leq b \wedge g$, the claim follows once it is shown that $b \wedge g \leq d_i$. If $d_i(j) = g(j)$, then

$$(b \wedge g)(j) = \min\{b(j), g(j)\} \leq g(j) = d_i(j).$$

If $d_i(j) < g(j)$, then $x_j \notin Z_{d_i}$ and hence $b(j) = p(j)$. Together with the inequality $p(j) \leq d_i(j) < g(j)$, we obtain that $(b \wedge g)(j) = p(j) \leq d_i(j)$. In both cases the claim follows.

Similarly, since $x^b \in x^q K[Z_{d_j}]$ we see that $b \wedge g \in [c_j, d_j]$. This is a contradiction, since $[c_i, d_i] \cap [c_j, d_j] = \emptyset$.

The statement about the Stanley depth of $\mathscr{D}(\mathscr{P})$ follows immediately from the the definitions.

(b) Let $\mathscr{D}$ be an arbitrary Stanley decomposition of I/J. First, to each $b \in P_{I/J}^g$ we assign an interval $[c, d] \subset P_{I/J}^g$: since $x^b \in I \setminus J$, there exists a Stanley space $x^c K[Z]$ in the decomposition $\mathscr{D}$ of I/J with $x^b \in x^c K[Z]$. It follows that $c \in P_{I/J}^g$ and $b(j) = c(j)$ for all j with $x_j \notin Z$. Now, we define $d \in \mathbb{N}^n$ by setting

$$d(j) = \begin{cases} g(j), & \text{if } x_j \in Z, \\ c(j), & \text{if } x_j \notin Z. \end{cases}$$

Observe that $[c, d] \subset P_{I/J}^g$. We noticed already that $c \in P_{I/J}^g$. It remains to be shown that $d \in P_{I/J}^g$. Since $x^c K[Z] \in I \setminus J$, it follows that $x^{c + \sum_j n_j \varepsilon_j} \in I \setminus J$, where the sum is taken over all j with $x_j \in Z$ and where for all j we have $n_j \in \mathbb{Z}_{\geq 0}$. Therefore $d = c + \sum_j (g(j) - c(j))\varepsilon_j \in P_{I/J}^g$.

Next we show that $b \in [c, d]$. For this we need to show that $b \leq d$. Indeed, if $x_j \in Z$, then $b(j) \leq g(j) = d(j)$. Otherwise $d(j) = c(j) = b(j)$ and consequently the inequality holds. Since $b \in [c, d]$, we obtain that $x^b \in x^c K[Z_d]$, and $Z \subseteq Z_d$, according to the definition of d.

In order to complete the proof of our theorem we now show that the intervals constructed above provide a partition $\mathscr{P}$ of $P^g_{I/J}$ and that $\operatorname{sdepth}\mathscr{D}(\mathscr{P}) \geq \operatorname{sdepth}\mathscr{D}$.

It is clear that these intervals cover $P^g_{I/J}$. Therefore it is enough to check that for any $b_1, b_2 \in P^g_{I/J}$ with $b_1 \neq b_2$, the corresponding intervals obtained from our construction, say $[c_1, d_1]$ and $[c_2, d_2]$, satisfy either $[c_1, d_1] = [c_2, d_2]$ or $[c_1, d_1] \cap [c_2, d_2] = \emptyset$.

To each c_i corresponds a Stanley space $x^{c_i}K[Z_i]$ in the given Stanley decomposition $\mathscr{D}$. We consider two cases. In the first case, we assume that $c_1 = c_2$. Then $Z_1 = Z_2$, and consequently $d_1 = d_2$. Hence $[c_1, d_1] = [c_2, d_2]$. In the second case, we assume $c_1 \neq c_2$. In this case we prove that $[c_1, d_1] \cap [c_2, d_2] = \emptyset$. Assume, by contradiction, that there exists $e \in P^g_{I/J}$ such that $e \in [c_1, d_1] \cap [c_2, d_2]$. It follows from the construction of the interval $[c_1, d_1]$ that $c_1(j) = d_1(j)$ if $x_j \notin Z_1$. Therefore, $e \in [c_1, d_1]$ implies that $e(j) = c_1(j)$, for all j with $x_j \notin Z_1$, and hence we obtain that $x^e \in x^{c_1}K[Z_1]$. Analogously, one obtains that $x^e \in x^{c_2}K[Z_2]$, a contradiction since $x^{c_1}K[Z_1] \cap x^{c_2}K[Z_2] = 0$.

To establish now the inequality $\operatorname{sdepth}\mathscr{D}(\mathscr{P}) \geq \operatorname{sdepth}\mathscr{D}$, we observe that $\operatorname{sdepth}\mathscr{D}(\mathscr{P})$ is equal to the minimum of all integers $|Z_d|$ where $[c,d]$ belongs to $\mathscr{P}$. On the other hand, we already showed that for each Stanley space $x^cK[Z]$ in $\mathscr{D}$ such that $c \in P^g_{I/J}$ we have that $|Z_d| \geq |Z|$. This yields the desired inequality. □

We consider two examples to illustrate Theorem 2.1. As a first example, consider the partition of the poset $P_{\mathfrak{m}}$ given in Figure 2. According to Theorem 2.1 the Stanley decomposition corresponding to this partition is exactly the one given in (1).

The second example is displayed in Figure 3. In the first picture the hatched region corresponds to the K-vector space spanned by all monomials in $I \setminus J$ where $I = (x_1^2x_2^4, x_1^3x_2^3, x_1^5x_2)$ and $J = (x_1^4x_2^5, x_1^6x_2^2)$. The second picture shows a partition of $P^g_{I/J}$ where $g = (7,6)$. The partition is the following:

$$P^g_{I/J} = [(2,4),(3,6)] \cup [(4,3),(5,4)] \cup [(5,1),(7,1)] \cup [(3,3),(3,3)] \cup [(5,2),(5,2)].$$

To this partition corresponds by Theorem 2.1 the following Stanley decomposition

$$I/J = (x_1^2x_2^4K[x_2] \oplus x_1^3x_2^4K[x_2]) \oplus (x_1^4x_2^3K \oplus x_1^5x_2^3K \oplus x_1^4x_2^4K \oplus x_1^5x_2^4K) \oplus x_1^5x_2K[x_1] \oplus x_1^3x_2^3K \oplus x_1^5x_2^2K$$

which is shown in the third picture of Figure 3.

If we want to use Theorem 2.1 in concrete cases to compute the Stanley depth, it is advisable to choose g such that the poset $P^g_{I/J}$ is as small as possible. If $G(I) = \{x^{a_1}, \ldots, x^{a_r}\}$ and $G(J) = \{x^{b_1}, \ldots, x^{b_s}\}$, then with $g = a_1 \vee \cdots \vee a_r \vee b_1 \vee \cdots \vee b_s$ the poset $P^g_{I/J}$ has the least number of elements and it is denoted by $P_{I/J}$. Even in this case, when $P_{I/J}$ has the least number of elements, it is very hard to compute in general $\operatorname{sdepth} I/J$, even though it can be done in a finite number of steps. A priori the strategy for computing $\operatorname{sdepth} I/J$ consists in two steps: 1) first one has to find a k, with $k \in \{0, \ldots, n\}$ such that for every partition $\mathscr{P}$ of $P_{I/J}$ we have $\operatorname{sdepth}\mathscr{D}(\mathscr{P}) < k+1$; 2) then one has to find a partition $\mathscr{P}$ of $P_{I/J}$ such that $\operatorname{sdepth}\mathscr{D}(\mathscr{P}) = k$. With these two steps fulfilled one gets that $\operatorname{sdepth} I/J = k$. The second step is in general quite hard and can lead to very difficult combinatorial problems.

To exemplify this consider the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ in $S = K[x_1, \ldots, x_n]$ and try to compute $\operatorname{sdepth}\mathfrak{m}$. A priori it seems to be the "easiest" monomial ideal in n variables

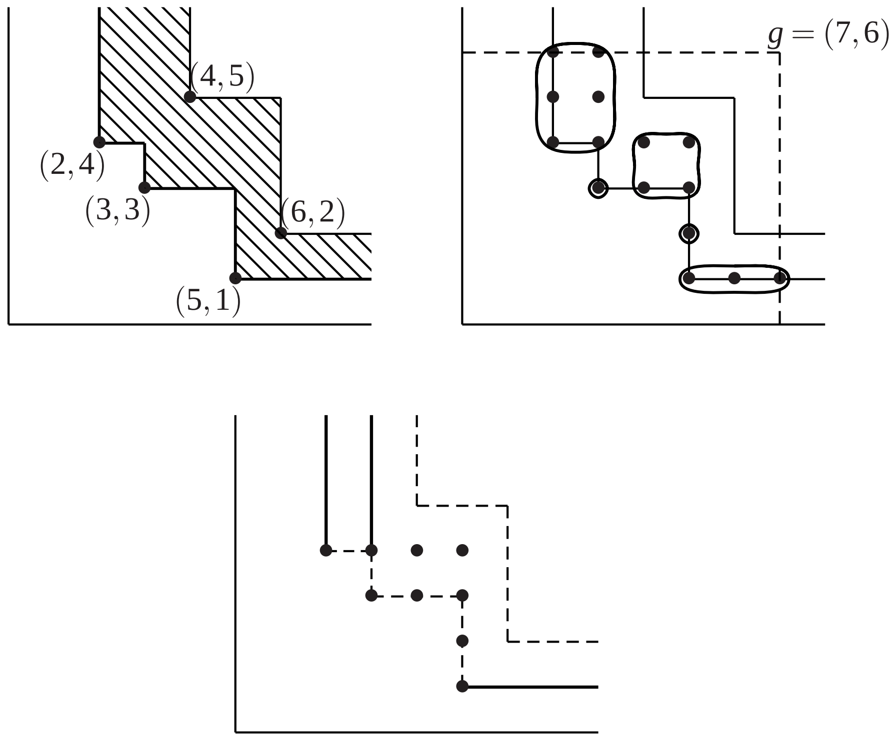

FIGURE 3

that one can consider and we should expect that the computation of its sdepth can be done relatively easy.

Let us consider first the case when $n=4$. Then $P_{\mathfrak{m}}$ is the following set

$$\begin{array}{cccccc} & 1000 & 0100 & 0010 & 0001 & \\ 1100 & 1010 & 1001 & 0110 & 0101 & 0011 \\ & 1110 & 1101 & 1011 & 0111 & \\ & & & 1111 & & \end{array}$$

Let $A=[1000,1100]\cup[0100,0110]\cup[0010,0011]\cup[0001,1001]$. Then $A\cup\bigcup_{a\in P_{\mathfrak{m}}\setminus A}[a,a]$ is a partition of $P_{\mathfrak{m}}$ and by Theorem 2.1 we obtain that $\operatorname{sdepth}\mathfrak{m}\geq 2$. On the other hand, since $\mathfrak{m}$ is not principal we have $\operatorname{sdepth}\mathfrak{m}\leq 3$. Assume that $\operatorname{sdepth}\mathfrak{m}=3$. By Theorem 2.1 there exists a partition of $P_{\mathfrak{m}}$ into disjoint intervals such that the end point of each interval has at least 3 of 1 (in this poset for any $a\in P_{\mathfrak{m}}$, $|Z_a|$ equals the number of 1's that appear in a). If one of these intervals is $[\varepsilon_i,1111]$, say $[1000,1111]$, then one of the intervals $[\varepsilon 2,0111]$, $[\varepsilon_3,0111]$, $[\varepsilon_4,0111]$ would have to cover the rest, a contradiction. Otherwise we have four disjoint intervals of type $[a,b]$, where $a\in\{1000,0100,0010,0001\}$ and b runs over the set $\{1110,1101,1011,0111\}$. Therefore the number of elements in $P_{\mathfrak{m}}$ with two 1's is at least $4\times 2=8$, a contradiction. Hence, our assumption is false and consequently $\operatorname{sdepth}\mathfrak{m}=2=\lceil 4/2\rceil$.

Now, for arbitrary n looking at the poset $P_{\mathfrak{m}}$ it is easy to see, by a similar counting argument, that $\operatorname{sdepth}\mathfrak{m}\leq\lceil n/2\rceil$, where $\lceil n/2\rceil$ denotes the smallest integer $\geq n/2$. To show equality one has to find a partition $\mathscr{P}$ of the poset $P_{\mathfrak{m}}$ such that $\operatorname{sdepth}\mathscr{D}(\mathscr{P}) = \lceil n/2 \rceil$. The equality was conjectured in [8] and was proved (not easy!) by Biro&al. by explicitly constructing such a partition.

Theorem 2.2. *(*[4]*) Let* $\mathfrak{m} = (x_1, \dots, x_n)$ *be the maximal ideal of* $S = K[x_1, \dots, x_n]$. *Then* $\operatorname{sdepth}\mathfrak{m} = \lceil n/2 \rceil$.

Using this result Shen was able to extend it and to compute the sdepth of an arbitrary complete intersection monomial ideal. Again, the proof is not easy at all. More precisely, he showed that

Theorem 2.3. *(*[13]*) Let* $I \subset S = K[x_1, \dots, x_n]$ *be a complete intersection monomial ideal minimally generated by* m *elements. Then* $\operatorname{sdepth} I = n - \lfloor m/2 \rfloor$, *where* $\lfloor m/2 \rfloor$ *denotes the biggest integer* $\leq \lfloor m/2 \rfloor$.

The next result clarifies for which partitions $\mathscr{P}$ of $P^g_{I/J}$ the Stanley decomposition $\mathscr{D}(\mathscr{P})$ of I/J is induced by a prime filtration and shows that $\operatorname{fdepth} I/J$ can be computed in a finite number of steps. Recall that a subset S of a poset P is called an *order filter* if for all $x \in S$ and all $y \geq x$ one has $y \in S$ as well. Then, similarly as for the sdepth Herzog&al. proved that

Theorem 2.4. *(*[8]*)* (a) *Let* $\mathscr{P}: P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ *be a partition of* $P^g_{I/J}$ *with the property that for all* j *the union* $\bigcup_{i=1}^j [c_i, d_i]$ *is an order filter in* $P^g_{I/J}$. *Then* $\mathscr{D}(\mathscr{P})$ *is induced by a prime filtration.*

(b) *Let* $\mathscr{D}$ *be a Stanley decomposition of* I/J *induced by a prime filtration of* I/J. *Then there exists a partition* $\mathscr{P}$ *of* $P^g_{I/J}$ *with the property that* $\mathscr{D}(\mathscr{P})$ *is induced by a prime filtration and such that* $\operatorname{sdepth}\mathscr{D}(\mathscr{P}) \geq \operatorname{sdepth}\mathscr{D}$.

In particular $\operatorname{fdepth} I/J$ *is the maximum of the numbers* $\operatorname{sdepth}\mathscr{D}(\mathscr{P})$, *where the maximum is taken over all partitions* $\mathscr{P}: P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ *of* $P^g_{I/J}$ *with the property that for all* j, *the union* $\bigcup_{i=1}^j [c_i, d_i]$ *is an order filter in* $P^g_{I/J}$.

Now we consider some applications of the algorithms for computing the sdepth and fdepth of I/J given by Theorem 2.1 and Theorem 2.4. As a first application of Theorem 2.4 one can read from the poset $P^g_{I/J}$ the Krull dimension of I/J.

Corollary 2.5. *(*[8]*)* $\dim I/J = \max\{|Z_c| : c \in P^g_{I/J}\}$.

Proof. For any prime filtration $\mathscr{F}$ of I/J we have that $\dim I/J = \max\{\dim S/P : P \in \operatorname{supp}(\mathscr{F})\}$. Now consider the canonical partition $\mathscr{P}: P^g_{I/J} = \bigcup_{c \in P^g_{I/J}} [c, c]$. We choose a total order $\succ$ of the intervals with the property that $[c, c] \succ [d, d]$ implies that $|d| \leq |c|$. Then the union of any initial sequence of these intervals is an order filter in $P^g_{I/J}$. Therefore it follows from Theorem 2.4 that $\mathscr{D}(\mathscr{P}) : I/J = \bigoplus_{c \in P^g_{I/J}} x^c K[Z_c]$ is induced by a prime filtration $\mathscr{F}'$ of I/J.

It follows that

$$\dim I/J = \max\{\dim S/P : P \in \operatorname{supp}(\mathscr{F}')\} = \max\{|Z_c| : c \in P^g_{I/J}\}.$$

□

In the example illustrated by Figure 3 we obtain that $\dim I/J = 1$ since $|Z_c| = 1$ for $c \in \{(2,6),(3,6),(7,1)\}$ and 0 otherwise.

We now give a lower bound for the sdepth of a monomial ideal by using the algorithm for the computation of the sdepth given by Theorem 2.1 together with induction on the number of variables. Let $I \subset S$ be a monomial ideal with $G(I) = \{x^{a_1},\dots,x^{a_m}\}$. We set $a = a_1 \vee a_2 \vee \cdots \vee a_m$. Then we can write P_I as a disjoint union $P_I = \bigcup_{j=p}^{q} A_j$, where $p = \min\{a_1(n),\dots,a_m(n)\}$, $q = a(n)$ and $A_j = \{c \in P_I : c(n) = j\}$. For all j with $p \le j \le q$ we let I_j be the monomial ideal of $K[x_1,\dots,x_{n-1}]$ such that $I \cap {x_n}^j K[x_1,\dots,x_{n-1}] = {x_n}^j I_j$. Then for all j with $p \le j \le q$, we have $A_j = \{(c,j) : c \in P_{I_j}^g\}$ with $g = (a(1),\dots,a(n-1))$.

Proposition 2.6. *(*[8]*) With the notation introduced we have*

$$\operatorname{sdepth} I \ge \min\{\operatorname{sdepth} I_p,\dots,\operatorname{sdepth} I_{q-1},\operatorname{sdepth} I_q + 1\}.$$

Proof. By Theorem 2.1 there exists for each $j \in \{p,\dots,q\}$ a partition $P_{I_j}^g = \bigcup_{k=1}^{r_j}[c_{jk},d_{jk}]$ of $P_{I_j}^g$ with $\operatorname{sdepth} I_j = \min\{|Z_{d_{jk}}| : \ k = 1,\dots,r_j\}$. Since P_I is the disjoint union of the A_j it follows that $P_I = \bigcup_{j=p}^{q}\bigcup_{k=1}^{r_j}[(c_{jk},j),(d_{jk},j)]$ is a partition of P_I. We have

$$|Z_{(d_{jk},j)}| = \begin{cases} |Z_{d_{jk}}|, & \text{if } j < q, \\ |Z_{d_{jk}}| + 1, & \text{if } j = q. \end{cases}$$

Hence the conclusion follows from Theorem 2.1. □

We may also prove

Proposition 2.7. *(*[8]*) Let $I \subset S$ be a monomial ideal generated by m elements. Then*

$$\operatorname{sdepth} I \ge \max\{1, n-m+1\}$$

Proof. We may assume that m is the number of minimal monomial generators of I. Then we proceed by induction on n. If $n = 1$, then $I = (u)$ is a principal ideal with Stanley decomposition $I = uK[x_1]$. Therefore, $\operatorname{sdepth} I = 1$. For the induction step we shall use Proposition 2.6. Indeed, we already have that I_j is a monomial ideal of $K[x_1,\dots,x_{n-1}]$ for all j, with $p \le j \le q$. In addition, one can easily see that $|G(I_j)| < m$ for all j such that $j < q$, and $|G(I_q)| \le m$. Hence, by induction hypothesis we have $\operatorname{sdepth} I_j \ge \max\{1, n - |G(I_j)|\} \ge \max\{1, n-m+1\}$ for all j with $j < q$, and similarly the induction hypothesis implies that $\operatorname{sdepth} I_q \ge \max\{1, n-m\}$, so that $\operatorname{sdepth} I_q + 1 \ge \max\{2, n-m+1\} \ge \max\{1, n-m+1\}$. Applying now Proposition 2.6 we obtain the desired inequality. □

Lemma 2.8. *(*[8]*) Let $J \subset I$ be monomial ideals of S, and let $T = S[x_{n+1}]$ be the polynomial ring over S in the variable x_{n+1}. Then*

$$\operatorname{depth} IT/JT = \operatorname{depth} I/J + 1, \quad \operatorname{fdepth} IT/JT = \operatorname{fdepth} I/J + 1,$$
$$\operatorname{sdepth} IT/JT = \operatorname{sdepth} I/J + 1.$$

Proof. The statement about the depth is obvious since x_{n+1} is regular on IT/JT. In order to prove the other two equations we consider the characteristic poset $P_{I/J}$ of I/J as well as the characteristic poset $P_{IT/JT}$ of IT/JT. The map $P_{I/J} \to P_{IT/JT}$, $c \mapsto c^* = (c(1),\dots,c(n),0)$ is an isomorphism of posets with the additional property that $\rho(c) = \rho(c^*) - 1$. In particular, if $\mathscr{P}: \ P_{I/J} = \bigcup_{i=1}^{r}[c_i,d_i]$ is a partition of $P_{I/J}$, then $\mathscr{P}^*: \ P_{IT/JT} =$

$\bigcup_{i=1}^{r}[c_i^*, d_i^*]$ is a partition of $P_{IT/JT}$, and the assignment $\mathscr{P} \mapsto \mathscr{P}^*$ establishes a bijection between partitions of $P_{I/J}$ and $P_{IT/JT}$. Since $|Z_{d_i}| = |Z_{d_i^*} - 1$ we see that $\operatorname{sdepth} \mathscr{D}(\mathscr{P}) = \operatorname{sdepth} \mathscr{D}(\mathscr{P}^*) - 1$ for all partitions $\mathscr{P}$ of $P_{I/J}$. Therefore the desired equations follow from Theorem 2.1 and Theorem 2.4. □

REFERENCES

[1] J. Apel, On a conjecture of R. P. Stanley; Part I - Monomial Ideals, J. of Alg. Comb. **17**, (2003), 39–56.

[2] J. Apel, On a conjecture of R. P. Stanley; Part II - Quotients Modulo Monomial Ideals, J. of Alg. Comb. **17**, (2003), 57–74.

[3] I. Anwar and D. Popescu, Stanley conjecture in small embedding dimension, J. Alg. **318**(2007), 1027–1031.

[4] C. Biro, D. Howard, M. Keller, W. Trotter and S. Young, Interval partitions and Stanley depth, preprint 2008.

[5] A. Dress, A new algebraic criterion for shellability, Beitrage zur Alg. und Geom., **34**(1), (1993), 45–55.

[6] J. Herzog, D. Popescu, Finite filtrations of modules and shellable multicomplexes, Manuscripta Math. **121** (2006), 385–410.

[7] J. Herzog, A. Soleyman-Jahan, S. Yassemi, Stanley decompositions and partitionable simplicial complexes, J. Algebr. Comb. **27**(2008), 113–125.

[8] J. Herzog, M. Vladoiu and X. Zheng, How to compute the Stanley depth of a monomial ideal, to appear in J. Alg.

[9] D. Maclagan and G. Smith, Uniform bounds on multigraded regularity, J. Algebr. Geom. **14**(2005), 137–164.

[10] S. Nasir, Stanley decompositions and localization, Bull. Math. Soc. Sc. Math. Roumanie **51**(99), no.2 (2008), 151–158.

[11] D. Popescu, Stanley depth of multigraded modules, Arxiv:Math. AC/0801.2632, to appear in J. Algebra.

[12] G. Rinaldo, An algorithm to compute the Stanley depth of monomial ideals, preprint 2008.

[13] Y. Shen, Stanley depth of complete intersection monomial ideals and upper-discrete partitions, J. Algebra **321** (2009), 1285–1292.

[14] A. Soleyman Jahan, Prime filtrations of monomial ideals and polarizations, J. Algebra, **312**(2007), 1011-1032, Arxiv:Math. AC/0605119.

[15] A. Soleyman Jahan, Stanley decompositions of squarefree modules and Alexander duality, arXiv:math.AC/0709.4145.

[16] R. P. Stanley, Linear Diophantine Equations and Local Cohomology, Invent. Math. **68** (1982), 175-193.