STANLEY DEPTH

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1. LECTURE: STANLEY DECOMPOSITIONS AND FILTRATIONS

Let $S = K[x_1,\dots,x_n]$ be a polynomial ring in n variables over a field K and M a finitely generated multigraded (i.e. $\mathbb{Z}^n$-graded) S-module. Given $m \in M$ a homogeneous element in M and $Z \subseteq \{x_1,\dots,x_n\}$, let $mK[Z] \subset M$ be the linear K-subspace of all elements of the form mf, $f \in K[Z]$. This subspace is called Stanley space of dimension $|Z|$, if $mK[Z]$ is a free $K[Z]$-module. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces $\mathscr{D}:\ M = \bigoplus_{i=1}^r m_iK[Z_i]$. Set $\operatorname{sdepth}\mathscr{D} = \min\{|Z_i| : i = 1,\dots,r\}$. The number

$$\operatorname{sdepth}(M) := \max\{\operatorname{sdepth}(\mathscr{D}) :\ \mathscr{D} \text{ is a Stanley decomposition of } M\}$$

is called Stanley depth of M. Some properties of Stanley depth appeared in [8], [6], [3], [2]. R. Stanley [9, Conjecture 5.1] gave the following conjecture.

Stanley's Conjecture $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ for all finitely generated $\mathbb{Z}^n$-graded S-modules M.

This lecture is completely based on [4]. We show here that the above conjecture holds when $\dim_S M \leq 2$ and $M = J/I$ for some monomial ideals $I \subset J$ of S with I square free. The result is true even when I is not square free (see [4]), but the proof is harder. If $n \leq 5$ Stanley's Conjecture holds for all cyclic S-modules by [1] and [4, Theorem 4.3]. We rely on the talks from this school of M. Vladoiu, where some preparations were made.

Let M be a finite multigraded S-module. A chain of multigraded submodules

$$\mathscr{F}:\ 0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

is called a *prime filtration* of M if $M_i/M_{i-1} \cong S/P_i(-a_i)$, where $a_i \in \mathbb{Z}^n$ and each P_i is a monomial prime ideal. We call the set $\{P_1,\dots,P_r\}$ the *support* of $\mathscr{F}$ and denote it $\operatorname{supp}\mathscr{F}$. Prime filtrations always exist and define Stanley decompositions as follows bellow. Suppose that the multigraded isomorphism $S/P_i(-a_i) \to M_i/M_{i-1}$ is given by $1 \to u_i + M_{i-1}$, where u_i is a $\mathbb{Z}^n$-homogeneous element of M_i of degree a_i. Set $Z_i = \{x_j : x_j \notin P_i\}$. Then

$$\mathscr{D}(\mathscr{F}):\ M = \oplus_{i=1}^r u_iK[Z_i]$$

is the Stanley decomposition of M induced by $\mathscr{F}$.

The above Stanley decomposition corresponds to the decomposition

$$M=\oplus_{i=1}^{r}M_i/M_{i-1}$$

as linear spaces. The Stanley decompositions induced by filtrations are too few and in general are not enough for computation of Stanley depth of M. However, given a filtration $\mathscr{F}$ we can define fdepth $\mathscr{F}=\min_{i\in[r]}\dim S/P_i$ and

$$\text{fdepth}(M):=\max\{\text{fdepth}(\mathscr{F}):$$
$$\mathscr{F}\text{ is a prime filtration of }M\}.$$

Clearly,

$$\text{fdepth }\mathscr{F}=\min_{i\in[r]}|Z_i|=\text{sdepth }\mathscr{D}(\mathscr{F})$$

and it follows that fdepth $M\leq$ sdepth M. Also note that

$$\text{depth }M\geq\min_{i\in[r]}\text{depth }M_i/M_{i-1}=$$
$$\min_{i\in[r]}\text{depth }S/P_i=\min_{i\in[r]}\dim S/P_i=\text{fdepth }\mathscr{F}.$$

The following three lemmas appeared in Vladoiu talks, and here we just remind them.

Lemma 1.1. fdepth $M\leq$ depth $M\leq$

$$\min\{\dim S/P:P\in\text{Ass }M\},$$

and fdepth $M\leq$ sdepth M. *If* $\dim_K M_a\leq 1$ *for all* $a\in\mathbb{Z}^n$ *then*

$$\text{sdepth }M\leq\min\{\dim S/P:P\in\text{Ass }M\}.$$

Lemma 1.2. *Suppose that M admits a prime filtration $\mathscr{F}$ with* supp $\mathscr{F}=$ Ass M *then* fdepth $M=$ depth $M=$

$$\min\{\dim S/P:P\in\text{Ass }M\}\leq\text{sdepth }M.$$

Moreover if $\dim_K M_a\leq 1$ *for all* $a\in\mathbb{Z}^n$ *then* fdepth $M=$ depth $M=$

$$\min\{\dim S/P:P\in\text{Ass }M\}=\text{sdepth }M.$$

M is *clean* if there exists a filtration $\mathscr{F}$ of M with supp $\mathscr{F}=$ Min M.

Lemma 1.3. *If M is a clean module then* fdepth $M=$ depth $M\leq$ sdepth M.

Next lemma is known for all finitely generated multigraded S-modules, but here we present only the case when M is reduced.

Lemma 1.4. *Let M be a finitely generated multigraded S-module with* $\text{Ass}\,M=\{P_1,\dots,P_r\}$, $\dim S/P_i=1$ *for* $i\in[r]$. *Let* $0=\cap_{i=1}^{r}N_i$ *be an irredundant primary decomposition of* (0) *in M and suppose that* $P_i=\text{Ann}(M/N_i)$ *for all i. Then M is clean.*

Proof. Apply induction on r. If $r=1$ then M is torsion-free over S/P_1 and we get M free over S/P_1 since $\dim(S/P_1)=1$. Thus M is clean over S.

Suppose that $r>1$. Then $0=\cap_{i=1}^{r-1}(N_i\cap N_r)$ is an irredundant primary decomposition of (0) in N_r and by induction hypothesis we get N_r clean. Also M/N_r is clean because $\text{Ass}(M/N_r)=\{P_r\}$ (case $r=1$). Hence the filtration $0\subset N_r\subset M$ can be refined to a clean filtration of M. □

Next we will extend the above lemma for some reduced Cohen-Macaulay multigraded modules of dimension 2 (from now on we suppose that $n > 2$). But first we need some preparations.

Lemma 1.5. *Let $I \subset U$ be two monomial ideals of S such that* Ass S/I *contains only prime ideals of dimension* 2*, and* Ass S/U *contains only prime ideals of dimension* 1*. Suppose that I is reduced. Then there exists $p \in$* Ass S/I *such that $U/(p \cap U)$ is a Cohen-Macaulay module of dimension* 2*.*

Proof. Let $U = \cap_{j=1}^{t} Q_j$ be a reduced primary decomposition of U and $P_j = \sqrt{Q_j}$. Since $P_1 \supset I$ there exists a minimal prime ideal p_1 containing I such that $P_1 \supset p_1$. Thus $P_1 = (p_1, x_k)$ for some $k \in [n]$. We may suppose that $p_1 = (x_2, \ldots, x_{n-1})$ and $k = n$ after renumbering the variables. Using the description of monomial primary ideals we note that $Q_1 + p_1 = (p_1, x_n^{s_1})$ for some positive integer s_1. Let

$$\mathscr{I} = \{i \in [t] : P_i = (p_i, x_n) \text{ for some } p_i \in \text{Min } S/I\}.$$

Clearly, $1 \in \mathscr{I}$. If $i \in \mathscr{I}$, that is $P_i = (p_i, x_n)$, then as above $Q_i + p_i = (p_i, x_n^{s_i})$ for some positive integers s_i. We may suppose that $s_1 = \max_{i \in \mathscr{I}} s_i$. Then we claim that depth $S/(p_1 + U) = 1$.

Let $r \in [t]$ be such that $p_1 + Q_r$ is m-primary. Since

$$U = U + I = \cap_{j \in [t], q \in \text{Ass } S/I} (q + Q_j)$$

and depth $S/U = 1$ we see that $p_1 + Q_r$ contains an intersection $\cap_{j=1}^{e} (q_j + Q_{c_j})$ with $q_j \in$ Ass S/I, $c_j \in [t]$ and $\dim(q_j + Q_{c_j}) = 1$, that is $q_j \subset P_{c_j}$. Note that $p_1 + Q_r =$

$$p_1 + (x_1^{a_1}, x_n^{a_n}, \text{ some } x^e \text{ with supp } e = \{1, n\}).$$

Suppose that $x_n \in P_{c_j}$ for all $j \in [e]$. Then we show that $p_1 + Q_1 \subset p_1 + Q_r$. Indeed, by hypothesis a power of x_n belongs to the minimal system of generators of $q_j + Q_{c_j}$, let us say $x_n^{b_{c_j}} \in G(q_j + Q_{c_j})$. Set $b = \max_j b_{c_j}$. Then $x_n^b \in \cap_{j=1}^{e} (q_j + Q_{c_j}) \subset p_1 + Q_r$ and so $b \geq a_n$. Let $b = b_{c_j}$ for some j. If $x_n \in q_j$ then $b = 1$ and so $a_n = 1$ and clearly $p_1 + Q_1 \subset P_1 \subset p_1 + Q_r$. If $x_n \notin q_j$ then $c_j \in \mathscr{I}$ and so $b = s_{c_j} \leq s_1$. Thus $p_1 + Q_1 = (p_1, x_n^{s_1}) \subset (p_1, x_n^b) \subset p_1 + Q_r$.

Now suppose that there exists $j \in [e]$ such that $x_n \notin P_{c_j}$, let us say $j = 1$. Then $P_{c_1} = (x_1, \ldots, x_{n-1})$ and $x_n \in P_{c_j}$ for all $j > 1$. As above, let $x_n^{b_{c_j}} \in G(q_j + Q_{c_j})$ for $j > 1$ and $b = \max_{j>1}^{e} b_{c_j}$. If $b \geq a_n$ then as above $p_1 + Q_1 \subset p_1 + Q_r$. Assume $b < a_n$. Let x_1^d be the power of x_1 contained in $G(q_1 + Q_{c_1})$. If $d \geq a_1$ then we get $p_1 + Q_{c_1} = (p_1, x_1^d) \subset p_1 + Q_r$ and $\dim(p_1 + Q_{c_1}) = 1$. Suppose that $d < a_1$. Then note that $x_1^d x_n^b \in G(\cap_{j=1}^{e} (q_j + Q_{c_j}))$ and so $x_1^d x_n^b \in p_1 + Q_r$. Thus

$$(p_1 + Q_1) \cap (p_1 + Q_{c_1}) \subset (p_1, x_1^d x_n^b) \subset p_1 + Q_r.$$

Hence $p_1 + U$ is the intersection of primary ideals of dimension 1, that is depth $S/(p_1 + U) = 1$. From the exact sequence

$$0 \to U/(p_1 \cap U) \to S/p_1 \to S/(p_1 + U) \to 0$$

we get $\operatorname{depth} U/(p_1\cap U)=2$. □

Lemma 1.6. *Let $I\subset U$ be two monomial ideals such that U/I is a Cohen-Macaulay S-module of dimension 2 and $\operatorname{Ass} S/I$ contains only prime ideals of dimension 2. Suppose that I is reduced. Then there exists $p\in \operatorname{Ass} S/I$ such that $U/(p\cap U)$ is a Cohen-Macaulay module of dimension 2.*

Proof. Let $U=\cap_{j=1}^t Q_j$ be a reduced primary decomposition of U and $P_j=\sqrt{Q_j}$. From the exact sequence

$$0\to U/I\to S/I\to S/U\to 0$$

we get $\operatorname{depth} S/U\geq$

$$\min\{\operatorname{depth} U/I-1,\operatorname{depth} S/I\}\geq 1.$$

Thus $1\leq \dim S/P_j\leq 2$ for all j. Suppose that $\dim S/P_j=2$. Then we have $\Pi_{q\in \operatorname{Ass} S/I}\, q\subset I\subset U\subset Q_j\subset P_j$. Thus $P_j\supset p$ for some $p\in \operatorname{Ass} S/I$ and we get $P_j=p$ because $\dim S/P_j=\dim S/p$. It follows that $\Pi_{q\in \operatorname{Ass} S/I, q\neq p} q\not\subset P_j$ and so $p\subset Q_j\subset P_j=p$. Hence $p=Q_j$. Set $U'=\cap_{i=1,i\neq j}^t Q_i$, $I'=\cap_{q\in \operatorname{Ass} S/I, q\neq p}\, q$. Then

$$(U+I')/I'\cong U/(U\cap I')=U/(U'\cap p\cap I')=$$
$$U/(U'\cap I)=U/I.$$

Changing I by I' and U by $U+I'$ we may reduce to a smaller $|\operatorname{Ass} S/I|$. By recurrence we may reduce in this way to the case when $\dim S/Q_j=1$ for all $j\in[t]$ since $I\neq U$. Now is enough to apply the above lemma. □

The above lemma cannot be extended to show that $U/(p\cap U)$ is Cohen-Macaulay for all $p\in \operatorname{Ass} S/I$, as shows the following:

Example 1.7. Let

$$I=(x_1x_2)\subset U=(x_1,x_3^2)\cap(x_2,x_3)$$

be monomial ideals of $S=K[x_1,x_2,x_3]$. We have $\operatorname{depth} S/U=1$ and $\operatorname{depth} S/I=2$. Thus U/I is Cohen-Macaulay but $U/(U\cap(x_2))$ is not since $U+(x_2)=(x_1,x_2,x_3^2)\cap(x_2,x_3)$, that is $\operatorname{depth} S/(U+(x_2))=0$. However, $U/(U\cap(x_1))$ is Cohen-Macaulay because $U+(x_1)=(x_1,x_3^2)$.

Theorem 1.8. *Let $I\subset U$ be two monomial ideals such that U/I is a Cohen-Macaulay S-module of dimension 2. Suppose that I is reduced. Then U/I is clean. In particular* $\operatorname{fdepth}\, U/I=\operatorname{sdepth}\, U/I=2$.

Proof. We follow the second part of the proof of Lemma 1.3. Let $I=\cap_{i=1}^r p_i$ be a reduced primary decomposition of I (so p_i are prime ideals). Apply induction on r. If $r=1$ then U/I is a maximal Cohen-Macaulay (so free) over S/p_1. Thus U/I is clean. Suppose that $r>1$. Then there exists $j\in[r]$ such that $U/(p_j\cap U)$ is a Cohen-Macaulay module of dimension 2 using Lemma 1.6. From the exact sequence

$$0\to(p_j\cap U)/I\to U/I\to U/(p_j\cap U)\to 0$$

we see that $(p_j\cap U)/I$ is a Cohen-Macaulay module of dimension 2. Set $I'=\cap_{i=1,i\neq j}^r p_i$. We have $(p_j\cap U)/I\cong((p_j\cap U)+I')/I'$ because $(p_j\cap U)\cap I'=U\cap I=I$. Applying induction hypothesis we get the modules $((p_j\cap U)+I')/I'$ and $(U+p_j)/p_j\cong U/(p_j\cap U)$ clean and so the filtration $0\subset(p_j\cap U)/I\subset U/I$ can be refined to a clean one. □

Theorem 1.9. *Let U, I be some monomial ideals of S such that $I \subset U$, $U \neq I$. If* $\dim_S U/I \leq 2$ *then* $\operatorname{sdepth}_S U/I \geq \operatorname{depth}_S U/I$.

Proof. If U/I is a Cohen-Macaulay S-module of dimension 2 then it is enough to apply the above theorem. If $\operatorname{depth}_S U/I = 1$ then the result follows from [4, Theorem 3.11]. □

2. Lecture: Weak Conjecture

In this lecture we study Stanley's Conjecture on monomial square free ideals of S, that is:

Weak Conjecture Let $I \subset S$ be a monomial square free ideal. Then $\operatorname{sdepth}_S I \geq \operatorname{depth}_S I$.

This conjecture says in fact that $\operatorname{sdepth}_S I \geq 1 + \operatorname{depth}_S S/I$ for any monomial square free ideal I of S. This remind us a question raised in [7], saying that $\operatorname{sdepth}_S I \geq 1 + \operatorname{sdepth}_S S/I$ for any monomial ideal I of S. A positive answer of this question in the frame of monomial square free ideals would state the Weak Conjecture as follows:

$$\operatorname{sdepth}_S I \geq 1 + \operatorname{sdepth}_S S/I \geq 1 + \operatorname{depth}_S S/I = \operatorname{depth}_S I,$$

the second inequality being a consequence of [4, Theorem 4.3], or of our Theorem 1.9.

First we present a result of Asia Rauf which we will need in the proof of our Lemma 2.8

Proposition 2.1. *([7])* $\operatorname{depth}_S S/(I, x_n) \geq \operatorname{depth}_S S/I - 1$.

We will need later also the following easy lemma:

Lemma 2.2. *Let $I \subset J$, $I \neq J$ be some monomial ideals of $S' = K[x_1, \ldots, x_{n-1}]$ and $T = (I + x_n J)S$. Then*

(1)
$$\operatorname{sdepth} T \geq \min\{\operatorname{sdepth}_{S'} I, \operatorname{sdepth}_S JS\},$$

(2)
$$\operatorname{sdepth} T \geq \min\{\operatorname{sdepth}_S JS/IS, \operatorname{sdepth}_S IS\}.$$

Proof. Note that $T = I \oplus x_n JS$ as linear K-spaces and so (1) holds. On the other hand the filtration $0 \subset IS \subset T$ induces an isomorphism of linear K-spaces $T \cong IS \oplus T/IS$ and so

$$\operatorname{sdepth} T \geq \min\{\operatorname{sdepth}_S T/IS, \operatorname{sdepth}_S IS\}.$$

Note that the multiplication by x_n induces an isomorphism of linear K-spaces $JS/IS \cong T/IS$, which shows that $\operatorname{sdepth}_S T/IS = \operatorname{sdepth}_S JS/IS$. Thus (2) holds too. □

It is the purpose of this section to study Stanley's Conjecture on monomial square free ideals of S, that is the Weak Conjecture.

Let $S' = K[x_1, \ldots, x_{n-1}]$ be a polynomial ring in $n-1$ variables over a field K and $U, V \subset S'$, $U \subset V$ two homogeneous ideals. We want to study the depth of the ideal $W = (U + x_n V)S$ of S. Actually every monomial square free ideal T of S has this form because then $(T : x_n)$ is generated by an ideal $V \subset S'$ and $T = (U + x_n V)S$ for $U = T \cap S'$.

Lemma 2.3. *Suppose that $U \neq V$ and* $\operatorname{depth}_{S'} S'/U = \operatorname{depth}_{S'} S'/V = \operatorname{depth}_{S'} V/U$. *Then* $\operatorname{depth}_S S/W = \operatorname{depth}_{S'} S'/U$.

Proof. Set $r = \operatorname{depth}_{S'} S'/U$ and choose a sequence $f_1, \dots, f_r$ of homogeneous elements of $m_{n-1} = (x_1, \dots, x_{n-1}) \subset S'$, which is regular on S'/U, S'/V and V/U simultaneously. Set $\bar{U} = (U, f_1, \dots, f_r)$, $\bar{V} = (V, f_1, \dots, f_r)$. Then tensorizing by $S'/(f_1, \dots, f_r)$ the exact sequence

$$0 \to V/U \to S'/U \to S'/V \to 0$$

we get the exact sequence

$$0 \to V/U \otimes_{S'} S'/(f_1, \dots, f_r) \to S'/\bar{U} \to S'/\bar{V} \to 0$$

and so $\bar{V}/\bar{U} \cong V/U \otimes_{S'} S'/(f_1, \dots, f_r)$ has depth 0.

Note that $f_1, \dots, f_r$ is regular also on S/W and taking $\bar{W} = W + (f_1, \dots, f_r)S$ we get $\operatorname{depth}_S S/W = \operatorname{depth}_S S/\bar{W} + r$. Thus passing from U, V, W to $\bar{U}, \bar{V}, \bar{W}$ we may reduce the problem to the case $r = 0$.

If $r = 0$ then there exists an element $v \in V \setminus U$ such that $(U : v) = m_{n-1}$. Thus the non-zero element of S/W induced by v is annihilated by m_{n-1} and x_n because $v \in V$. Hence $\operatorname{depth}_S S/W = 0$. □

Example 2.4. Let $n = 4$, $V = (x_1, x_2)$, $U = V \cap (x_1, x_3)$ be ideals of $S' = K[x_1, x_2, x_3]$ and $W = (U + x_4 V)S$. Then $\{x_3 - x_2\}$ is a maximal regular sequence on V/U and on S/W as well. Thus $\operatorname{depth}_{S'} V/U = \operatorname{depth}_{S'} S'/U = \operatorname{depth}_{S'} S'/V = \operatorname{depth}_S S/W = 1$.

Lemma 2.5. *Let $I, J \subset S'$, $I \subset J$, $I \neq J$ be two monomial ideals, $T = (I + x_n J)S$ such that*

(1) $\operatorname{depth}_{S'} S'/I = \operatorname{depth}_S S/T - 1$,
(2) $\operatorname{sdepth}_{S'} I \geq 1 + \operatorname{depth}_{S'} S'/I$,
(3) $\operatorname{sdepth}_{S'} J/I \geq \operatorname{depth}_{S'} J/I$.

Then $\operatorname{sdepth}_S T \geq 1 + \operatorname{depth}_S S/T$.

Proof. By Lemma 2.2 we have

$$\operatorname{sdepth}_S T \geq 1 + \min\{\operatorname{sdepth}_{S'} I, \operatorname{sdepth}_{S'} J/I\} \geq 1 + \min\{1 + \operatorname{depth}_{S'} S'/I, \operatorname{depth}_{S'} J/I\}$$

using (3), (2) and a Lemma of Herzog-Vladoiu-Zheng. Note that in the following exact sequence

$$0 \to S/JS = S/(T : x_n) \xrightarrow{x_n} S/T \to S/(T, x_n) \cong S'/I \to 0$$

we have $\operatorname{depth}_S S/JS = \operatorname{depth}_{S'} S'/I + 1$ because of (1) and the Depth Lemma. Thus $\operatorname{depth}_{S'} S'/I = \operatorname{depth}_{S'} S'/J$. As $\operatorname{depth}_{S'} S'/I \neq \operatorname{depth}_S S/T$ we get $\operatorname{depth}_{S'} S'/I \neq \operatorname{depth}_{S'} J/I$ by Lemma 2.3. But $\operatorname{depth}_{S'} J/I \geq \operatorname{depth}_{S'} S'/I$ because of the Depth Lemma applied to the following exact sequence

$$0 \to J/I \to S'/I \to S'/J \to 0.$$

It follows that $\operatorname{depth}_{S'} J/I \geq 1 + \operatorname{depth}_{S'} S'/I$ and so

$$\operatorname{sdepth}_S T \geq 2 + \operatorname{depth}_{S'} S'/I = 1 + \operatorname{depth}_S S/T.$$

□

Remark 2.6. The above lemma introduces the difficult hypothesis (3) and one can hope that it is not necessary at least for square free monomial ideals. It seems this is not the case as shows somehow the next example.

Example 2.7. Let $n=4$, $J=(x_1x_3,x_2)$, $I=(x_1x_2,x_1x_3)$ be ideals of $S'=K[x_1,x_2,x_3]$ and $T=(I+x_4J)S=(x_1,x_2)\cap(x_2,x_3)\cap(x_1,x_4)$. Then $\{x_4-x_2,x_3-x_1\}$ is a maximal regular sequence on S/T. Thus $\operatorname{depth}_S\ S/T=2$, $\operatorname{depth}_{S'}\ S'/I=\operatorname{depth}_{S'}\ S'/J=1$.

Lemma 2.8. *Let $I,J\subset S'$, $I\subset J$, $I\neq J$ be two monomial ideals, $T=(I+x_nJ)S$ such that*

(1) $\operatorname{depth}_{S'}\ S'/I\neq\operatorname{depth}_S\ S/T-1$,
(2) $\operatorname{sdepth}_{S'}\ I\geq 1+\operatorname{depth}_{S'}\ S'/I,\quad \operatorname{sdepth}_{S'}\ J\geq 1+\operatorname{depth}_{S'}\ S'/J.$

Then $\operatorname{sdepth}_S\ T\geq 1+\operatorname{depth}_S\ S/T$.

Proof. By Lemma 2.2 we have

$$\operatorname{sdepth}_S T\geq\min\{\operatorname{sdepth}_{S'}I,1+\operatorname{sdepth}_{S'}J\}\geq 1+\min\{\operatorname{depth}_{S'}S'/I,1+\operatorname{depth}_{S'}S'/J\}$$

using (2). Applying Proposition 2.1 we get $\operatorname{depth}_{S'}S'/I=\operatorname{depth}_S S/(T,x_n)\geq\operatorname{depth}_S S/T-1$, the inequality being strict because of (1). We have the following exact sequence

$$0\to S/JS=S/(T:x_n)\xrightarrow{x_n}S/T\to S/(T,x_n)\cong S'/I\to 0.$$

If $\operatorname{depth}_{S'}S'/I>\operatorname{depth}_S S/T$ then $\operatorname{depth}_S S/JS=\operatorname{depth}_S S/T$ by Depth Lemma and so

$$\operatorname{sdepth}_S T\geq 1+\min\{\operatorname{depth}_{S'}S'/I,\operatorname{depth}_S S/JS\}=1+\operatorname{depth}_S S/T.$$

If $\operatorname{depth}_{S'}S'/I=\operatorname{depth}_S S/T$ then $\operatorname{depth}_S S/JS\geq\operatorname{depth}_{S'}S'/I$ again by Depth Lemma and thus

$$\operatorname{sdepth}_S T\geq 1+\operatorname{depth}_{S'}S'/I=1+\operatorname{depth}_S S/T.$$

$\square$

Example 2.9. Let $n=5$, $J=(x_1,x_2,x_3)$, $I=(x_1,x_2)\cap(x_3,x_4)$ be ideals of $S'=K[x_1,\dots,x_4]$ and $T=(I+x_5J)S$. Then $\{x_4,x_3-x_1\}$ is a maximal regular sequence on J/I and so $\operatorname{depth}_{S'}\ J/I=2>1=\operatorname{depth}_{S'}\ S'/I=\operatorname{depth}_{S'}\ S'/J=\operatorname{depth}_S\ S/T$.

Theorem 2.10. *Suppose that the Stanley's conjecture holds for factors V/U of monomial square free ideals, $U,V\subset S'=K[x_1,\dots,x_{n-1}]$, $U\subset V$, that is* $\operatorname{sdepth}_{S'}\ V/U\geq\operatorname{depth}_{S'}\ V/U$. *Then the Weak Conjecture holds for monomial square free ideals of $S=K[x_1,\dots,x_n]$.*

Proof. Let $r\leq n$ be a positive integer and $T\subset S_r=K[x_1,\dots,x_r]$ a monomial square free ideal. By induction on r we show that $\operatorname{sdepth}_{S_r}\ T\geq 1+\operatorname{depth}_{S_r}\ S_r/T$, the case $r=1$ being trivial. Clearly, $(T:x_r)$ is generated by a monomial square free ideal $J\subset S_{r-1}$ containing $I=T\cap S_{r-1}$. By induction hypothesis we have $\operatorname{sdepth}_{S_{r-1}}\ I\geq 1+\operatorname{depth}_{S_{r-1}}\ S_{r-1}/I$, $\operatorname{sdepth}_{S_{r-1}}\ J\geq 1+\operatorname{depth}_{S_{r-1}}\ S_{r-1}/J$. If $I=J$ then $T=IS$, x_r is regular on S_r/T and we have

$$\operatorname{sdepth}_{S_r}\ T=1+\operatorname{sdepth}_{S_{r-1}}\ I\geq 2+\operatorname{depth}_{S_{r-1}}\ S_{r-1}/I=1+\operatorname{depth}_{S_r}\ S_r/T.$$

Now suppose that $I\neq J$. If $\operatorname{depth}_{S_{r-1}}\ S_{r-1}/I\neq\operatorname{depth}_{S_r}\ S_r/T-1$, then it is enough to apply Lemma 2.8. If $\operatorname{depth}_{S_{r-1}}\ S_{r-1}/I=\operatorname{depth}_{S_r}\ S_r/T-1$, then apply Lemma 2.5. $\square$

Corollary 2.11. *The Weak Conjecture holds in $S=K[x_1,\dots,x_4]$.*

Proof. It is enough to apply Lemmas 2.5, 2.8 after we show that for monomial square free ideals $I,J \subset S' = K[x_1,\dots,x_3]$, $I \subset J$, $I \neq J$, $T = (I + x_4J)S$ with $\operatorname{depth}_{S'} S'/I = \operatorname{depth}_S S/T - 1$, we have $\operatorname{sdepth}_{S'} J/I \geq \operatorname{depth}_{S'} J/I$. But then $I \neq 0$ because otherwise $\operatorname{depth}_S S/T \leq 3 = \operatorname{depth}_{S'} S'/I$, which is false. Thus $\dim_{S'} J/I \leq 2$ and we may apply Theorem 1.9. □

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