Problem Seminar

March 10, 2009: Continuity

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Continuous functions

1. Prove that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is Lipschitz, and therefore continuous.

2. Consider the function

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or if } y \geq x^2 \\ \sin \frac{\pi y}{x^2} & \text{if } 0 < y < x^2. \end{cases}$$

   Prove that f is not continuous at the origin (though its restriction to any straight line through the origin is continuous).

Questions marked with * are more involving.

3. Prove that the graph of any continuous function $f : [a, b] \longrightarrow \mathbb{R}$ is a closed subset of $\mathbb{R}^2$. Does the converse

   work?

4. Prove that every convex function defined on an open convex subset of $\mathbb{R}^n$ is continuous.

Uniform continuity

5. Consider a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\lim_{\|x\| \to \infty} f(x) = 0$. Prove that f is bounded and uniformly continuous.

6. A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called periodic if there exist two numbers $S, T > 0$ such that

$$f(x + S, y + T) = f(x, y) \text{ for all } (x, y) \in \mathbb{R}^2.$$

(a) Give examples of periodic functions of two variables.

(b) Prove that every continuous periodic function defined on $\mathbb{R}^2$ is uniformly continuous.

7. Prove that every uniformly continuous function can be extended by continuity to the closure of its domain of definition. Infer that the function $\sin(1/x)$ is not uniformly continuous on $(0, \infty)$.

The intermediate value theorem

8. Let $f : (0, 1) \to \mathbb{R}$ be a continuous function. Describe the range of f.

9. Is there any continuous one-to one function from the closed unit disc into the real line?

10. Consider a continuous real function f defined on the unit circle. Prove that f attains the same value at a pair of opposite points.

11*. A generalization of the intermediate value theorem is the *Brouwer fixed point theorem*: Every continuous map from a closed unit ball of $\mathbb{R}^N$ into itself has a fixed point. Infer from this result that the following system

$$\begin{aligned} 2x + 3y &= 1 + \frac{y^2 + xe^{1-\sin xy}}{1 + x^4 + y^2} \\ 3x - 2y &= 1 + \frac{x^2 - ye^{1+\sin xy}}{1 + x^2 + y^4} \end{aligned}$$

has at least four solutions.

References

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[3] C. P. Niculescu and L.-E. Persson, *Convex Functions and their Applications. A Contemporary Approach.* CMS Books in Mathematics **23**, Springer Verlag, 2006.

[4] M. H. Protter and C. B. Morrey, *A First Course in Real Analysis*, 2nd ed., Springer Verlag, 1991.

[5] W. Rudin: *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill Book Co., New York, 1976.