Problem Seminar

March 13, 2009: Mean value theorem. Taylor's formula. Convex functions

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Mean value theorem

1. Consider a differentiable function $f : D \to \mathbb{R}$ defined on an open subset of $\mathbb{R}^N$ and a pair of points $u, v \in D$ such that the segment $[u, v]$ lies in D. Prove that there is a point $w \in [u, v]$ such that

$$f(v) - f(u) = \langle \nabla f(w), v - u \rangle.$$

2. Prove that every differentiable function $f : D \to \mathbb{R}$ defined on a connected open subset of $\mathbb{R}^N$ is constant provided its gradient vanishes everywhere.

Questions marked with * are more involving.

Taylor's formula

3. Suppose that $f : \mathbb{R}^N \to \mathbb{R}$ and all its partial derivatives up to order $n+1$ are continuous on a ball $B_r(a)$. Prove that for each $x \in B_r(a)$ there is a point ξ on the line segment from a to x such that

$$f(x) = \sum_{|\alpha|\le n} D^\alpha f(a)(x-a)^\alpha + \sum_{|\alpha|=n+1} D^\alpha f(\xi)(x-a)^\alpha.$$

Convex functions

4. Prove that a twice differentiable function (defined on an open convex subset of $\mathbb{R}^N$) is convex if its Hessian matrix is positive.

5. Let $A = \{x_1 < x_2 < \cdots < x_N\} \subset \mathbb{R}^N$ and $\Delta_N(x) = \prod_{j<l}(x_j - x_l)$ be the Vandermonde determinant. Prove that $-\log \Delta_N$ is convex on A.

6. Prove that the function $f(x) = \log\left(\sum_{k=1}^{N} e^{x_k}\right)$ is convex on $\mathbb{R}^N$.

7. Prove that the function $f(x_1, ..., x_N) = \left(\prod_{k=1}^{N} x_k\right)^{1/N}$ is convex on

$$\mathbb{R}^N_{++} = \{x_1, ..., x_N \in \mathbb{R} : x_1, ..., x_N > 0\}.$$

References

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[4] M. H. Protter and C. B. Morrey, *A First Course in Real Analysis*, 2nd ed., Springer Verlag, 1991.

[5] W. Rudin: *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill Book Co., New York, 1976.