Problem Seminar

March 14, 2009: Inverse function theorem. Implicit functions

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Differentiation of implicit functions

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if

$$\left(x^2+y^2\right)^3-3\left(x^2+y^2\right)+1=0.$$

2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{dy}$ if

$$x^2-2y^2+3z^2-yz+y=0.$$

Questions marked with * are more involving.

3. Consider the Descartes' folium, $x^3 + y^3 - 3xy = 0$. Write down the equation of the tangent at the point $(3/2, 3/2)$.

4. Find the extrema of the function $z = z(x, y)$ defined by

$$2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0.$$

5. Given the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0, \end{aligned}$$

prove that it is possible to express (locally) y, z, u as functions of x. Notice that we cannot express x, y, z as functions of u.

6. Let E and F be two isomorphic Banach spaces, U an open subset of E and $f \in C^1(U, F)$ a function such that $df(x) \in \mathrm{Isom}\,(E, F)$ for every $x \in U$. Prove that $f(U)$ is an open subset of F.

References

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[4] M. H. Protter and C. B. Morrey, *A First Course in Real Analysis*, 2nd ed., Springer Verlag, 1991.

[5] W. Rudin: *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill Book Co., New York, 1976.