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BRILLIANT IDEAS OF COMBINATORICS

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Abstract

Main and beautiful ideas of combinatorics such as combinatorial reasoning, one-to-one correspondence, inclusion-exclusion principle, generating functions, and Bernside’s lemma will be discussed.

Introduction

“Mathematics is the queen of sciences”.

Using such a slogan we mean that mathematics studies laws of the nature in most common – abstract form. In such case for every part of mathematics we must have corresponding part of the nature.

What part of the nature corresponds to combinatorics?

Today my answer will be: “combinatorics in formal form reflects some laws of human mind”.

I give you some examples.

The first one is the investigation, which was done by famous mathematician **Jacques Hadamard** together with phsychologist **Rybo**. The results of this investigation were published in the beginnig of 20th century in the book **"*Psychology of Invention in the Mathematical Field*".**

They prepared a questionare for famous mathematicians. In the questionare they asked details of how they invent new results. In this book were describe many interesting cases of inventing new ideas. All mathematicians wrote that they have heavy work before insight came to them. But it is interesting that between hard work and the moment the new idea came was a quite period. In this preparatory period mathematicians did nothing, but their brain worked. Many mathematicians have wrote that in this period part of knowleges, ideas collide each other, new combinations borned, some of them were strange but some were most new and exiting.

So we can see that combinatorial processes are part of our mental processings.

Now I give two most famous examples of inventions where combinatorial reasonongs are most evident.

First principle of combinatorics: principle of multiplication

Example 1.

Dmitri Mendeleev (1834-1907) is a famous Russian chemist who arranged the 63 known elements into a periodic table based on atomic mass, which he published in Principles of Chemistry in 1869.

The wonderful legend says that Mendeleev invented the periodical system in his dream and this story is bases on a real fact told by Inostrantsev. Once Inostrantsev came to Mendeleev's private office and saw that Mendeleev was in a very gloomy mood. Mendeleev complained that he was thinking about the system and he almost got it in his mind but couldn't express in a table. Three days later he was still working at his favorite table in his office without having a rest for three nights trying to compile a table. Nothing helped so at the end he was very exhausted and fell asleep. In his dream he saw the table of elements very clear and when he woke up he wrote down everything he had seen and only one small correction was made afterwards.

Another legend adds to this story such details: when inventing periodic table he was prepared to a next lesson and searched for a good way to explain students' properties of chemical elements.

Now I think you know that one of the most basic principle of combinatorics – **law of multiplication**. If we want to classify objects having two characteristics we use table. For example, when we look at the car number LAH 5721 we imagine the table with two entrances: one for triples of letters (*ABC, …, XYZ*) and another for quadruples of digits (0000, …, 9999).

For example, if we use 9000 triples of letters and 10000 quadruples of digits we get 9000*10000=90 billion numbers.

The idea of multiplication principle is used in mathematics to denote set, which we get as a result of all combinations of pairs. For example, if we denote set of triples of letters as T, and set of quadruples of digits as Q, the set of all car numbers will be denoted as $T\text{x}Q$. Moreover if we denote alphabet as $A=\{a, b,\ldots,z\}$ and set of digits as $D=\{0,1,\ldots,9\}$, then set of car numbers will be denoted as

$$A\text{x}A\text{x}A\text{x}D\text{x}D\text{x}D\text{x}D \text{ or as } A^3D^4.$$

Later we shall return to the idea of a table.

But now next example.

Second principle: principle of addition

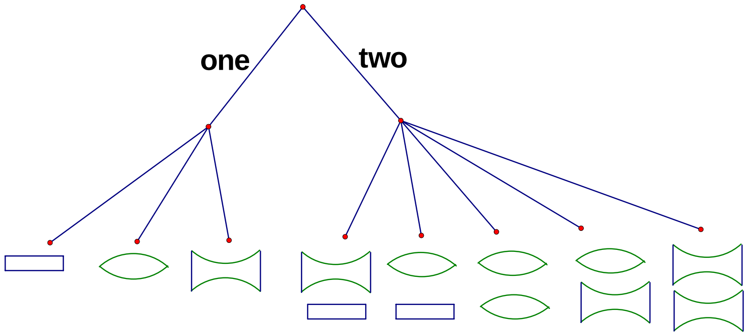
Example 2.

How Galileo reinvented the telescope.

Galileo learned about the telescope in 1609, but did not get a chance to see one (because it was commercial secret). So he decided to do it himself. He wrote: “My reasoning was this:

The device needs either a single glass or more than one. It cannot consist of one glass alone, because the shape of this would have to be convex (that is, thicker in the middle than at the edges) or concave (that is, thinner in the middle), or bounded by parallel surfaces. But the last-named does not alter visible objects in any way, either by enlarging or reducing them; the concave diminishes them; and the convex, though it does enlarge them, shows them indistinctly and confusedly. Passing then to two, and knowing as before that a glass with parallel faces alters nothing, I concluded that the effect would still not be achieved by combining such a glass with either of the other two. Hence I was restricted to discovering what would be done by a combination of the convex and the concave. You see how this gave me what I sought; and such were the steps in my discovery, in which I was assisted not at all by the received opinion that the goal was a real one.”

In this text we can see a tree represents typical combinatorial reasoning which usually.



The same tree we used for solving combinatorial problems.

This tree gives us second strong principle of combinatorics: **law of addition**.

We divide all combinations to some types, then count number of combinations of each kind and then add all results. This principle has a slogan **“divide and conquer”.**

For example. How many “words” can be constructed from alphabet with only three letters $\{a, c, t\}$ in such a way that every two letters staying beside be different.

Third principle: one-to-one correspondence

Now we shall consider a sample which shows us how various rearrangements and recombination can give us new knowledge.

Example

We are interested in enumerating all the possibilities to choose k objects from the given set of n different objects.

For example we have 5 objects. Name them $\{a, b, c, d, e\}$. We need to choose two of them. We have 10 possibilities:

ab, ac, ad, ae, bc, bd, be, cd, ce, de.

Now we need to choose three objects out of 5. We can do the same algorithm, but we can notice that if we consider the rest of chosen letters in the solution of first problem, we get solution of second problem.

Now if we name the number of ways to choose k object from the set of n objects as $C(n; k)$ we get the equality:

$$C(n; k) = C(n; n-k)$$

So you can see that we need not in any proof of this fact because we get one-to-one correspondence between two sets of combinations.

This is the third principle of combinatorics: to reduce one problem to another by using one-to-one correspondence. We shall meet this idea many times.

Many of students can say: it is not solution of the problem, because we don’t know how to compute $C(n; k)$.

So try to apply our idea of one-to-one correspondence to another problem.

Fourth principle: to compare different combinatorial reasoning

Problem.

We need to make a football team of k players (one of them must be a captain) from n candidates. How many ways we have?

First way of reasoning: choose k players from n candidates. We know that we have $C(n; k)$ ways for that. Then choose a captain from k players – k ways. Now use the multiplication principle to combine various combinations of team with different cases to choose a captain. We will receive $kC(n; k)$ possibilities.

Now try another way of reasoning: first of all let's choose the captain (n ways) and then choose the rest of the team - $C(n\text{-}1;k\text{-}1)$ ways. After applying multiplication law we will receive

$$nC(n-1;k-1).$$

We receive two expressions for one number. Hence:

$$kC(n; k) = nC(n-1;k-1).$$

After dividing by k we get recurrence formula:

$$C(n; k) = (n/k)C(n-1;k-1).$$

Due to that $C(n; k)$ is often denoted as C_k^n.

Now, for example, if n=5, k=3, we receive (5/3)(4/2)C(3;1). But value of C(3;1) is equal to 3 that is evident from definition of $C(n; k)$. Then C(5;3)=10.

Fifth principle: using of different languages (models)

Now before discussing next problem we try to interpret $C(n; k)$ to the language of words.

Let's encode the k-subset of our set with n elements in such a way. If element is chosen we encode it with 1, if not – with 0.

For example for the set $\{a, b, c, d, e\}$ subset $\{b, c, e\}$ is encoded by binary sequence 01101.

So we receive one-to-one correspondence between k-subsets of n-set and binary sequences with k ones and $(n\text{-}k)$ zeros.

Theorem.

The number of binary sequences with k ones and $(n\text{-}k)$ zeros is equal to $C(n; k)$.

Now consider the next problem.

Problem.

Confectioner's shop has 5 types of cakes. I want to by 10 cakes. How many ways I have for that?

From the first sight it is completely new problem. But try to find one-to-one correspondence with known set of combinations. Let’s encode every purchase with binary sequences. For example, sequence 00100010110000 mean that we buy 2 cakes of 1st kind, 3 of 2nd kind, 1 of 3rd kind, 0 of 4th kind, 4 of 5th kind. In other words, zero means a cake, and one means a boundary between different types of cakes.

Conversely, for every binary sequence with 10 zeroes and 4 ones we can point the particular purchase.

So solution of our problem is $C(14;4)=(14/4)(13/3)(12/2)(11/1)=7*13*11=1001$ (The same quantity as a number of Shehrezada nights. So instead of tales she can give 10 cakes in various combinations).

This result can be reformulated in abstract form:

Theorem.

The number of nonnegative integer solutions of an equation

$$X1+X2+\dots+Xk = n \text{ is equal to } C(n+k-1;k-1) = C(n+k-1;n)$$

Now you do the next step and try to understand how combinatorics transforms to algebraic structures.

Sixth principle: to operate formally with algebraic objects instead of combinatorial one

Let’s consider the next expression

$(x + y)(x + y)(x + y)\dots(x + y)$ n terms. We want to expand this expression and to simplify result.

To do this we will count all the terms which have k letters x and others $(n-k)$ letters y.

Each such a term can be encoded with binary sequence: 1 if we choose x from the particular brackets, and 0 otherwise (if we choose y).

So each term with k letters x and $(n-k)$ letters y $x^k y^{n-k}$ will be repeated $C(n; k)$ times.

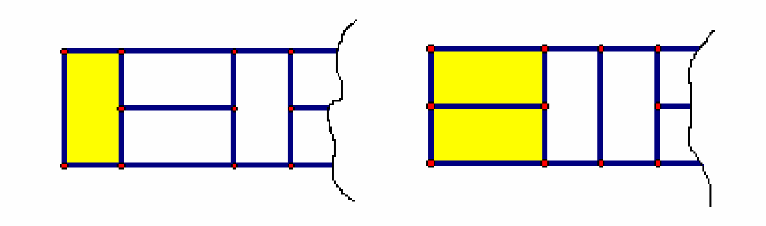
So we get formula:

$$(x + y)^n = x^n + C(n;n-1)x^{n-1} y+\dots+C(n; k)x^k y^{n-k} +\dots\dots\dots+y^n$$

Now we consider how these formal algebraic tools – polynomials – help us in solving combinatorial problems.

Problem.

We have strip with width 2 and length n (it means that we interested in method which give us answer for n=1, 2, 3 and so on). We will fill it with dominoes – rectangles 1x2 or 2x1 (we can use them in horizontal or vertical position).



We will try to arrange all the combinations for increasing value of n.
We can see that horizontal dominoes meet only by pairs. So if we denote vertical domino as x and double horizontal domino with y we can write an expression for every filling of the strip.

For example $xyxy$.., $yxxy$…

All the combinations can be written as infinite sum (series):

$$x+y+xx+xy+yx+\dots$$

Now we invent new interpretation for multiplication. We will consider multiplication of two rectangles of equal wideness as simple joining them in same order.

For example: $(xxy)(yx) = xxyyx$

Now we can rewrite the series in such a way

$$x(1+x+y+xx+xy+yx+\dots)+y(1+x+y+xx+xy+yx+\dots)$$

This is correct because if we cut first domino from all the fillings which begin with the vertical domino we get all the fillings again!

What means 1 in this sum? It is an “empty strip”. It means that joining such a strip to another one don’t change it.

So if we denote all the arrangements with the empty one as S we can write:

$$S=1+xS+yS$$

Using simple algebraic operations we get from this equation that

$$S=1/(1-x-y).$$

Now remember the formula for infinite arithmetic progression:

$$1/(1-z)=1+z+z^2+z^3+\ldots.$$

After substituting $(x + y)$ instead of z we will get

$$1/(1-x-y)=1+(x + y)+(x + y)^2+(x + y)^3+\ldots$$

How to interpret this result?

For example if we interested in all fillings of strip with 3 vertical and 4 (2 pairs) horizontal dominoes (n will be equal to 7) we must take a term $(x + y)^{3+2}$ and find the coefficient for x^3y^2. As we know it is binomial coefficient $C(5;3)=10$.

Now do the next step in revealing links between combinatorics and algebra.

The next idea gives us linkage between combinatorics and group theory.

Problem.

In how many ways we can color the sides of cube using 2 colors?

Try the idea with a table or tree: for one side we have 2 colors, then using multiplication law we get 2x2x2x2x2x2=64 for all six sides. Is it solution? No. Why? We can rotate cube and one coloring will transfer into another. So we need not to distinguish the colorings that are made from one another by cube rotation.

We have not enough time for cube then we consider instead of cube sides square sides.

We encode each coloring of square sides by binary sequences length 4.

For example 1010 mean that up and down sides are colored in 1st color, left and right sides – in 2nd (zero) color.

Now construct a table in which we will mark if the rotation of square change coloring or not.

	0	90	180	270
0000	+	+	+	+
0001	+			
0010	+			
0011	+			
0100	+			
0101	+		+	
0110	+			
0111	+			
1000	+			
1001	+			
1010	+		+	

1011	+			
1100	+			
1101	+			
1110	+			
1111	+	+	+	+

Now join in one group all the colorings we consider as same coloring. How many items we will have in each group? Exactly 4, which is equal to numbers of transformations? It is important result, which is named as Bernside's lemma.

Lemma.

If for every transformation which transform body to itself count the number of colorings, which are not changed, by this transformation, to add these numbers and to divide sum by the number of transformation we will receive the number of different colorings (orbits) for this body.

Now we discuss the new problem and new principle.

For square and two colors we will have 24/4=6 colorings.

For cube and two colors we will have

[3x2x(2x2x2)+3x1x(2x2x2x2)+1x(2x2x2x2x2x2)+6x(2x2x2)+4x2x(2x2)]:24=10

Seventh principle: inclusion-exclusion principle

Problem.

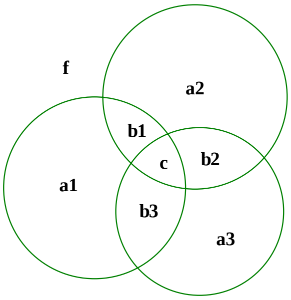
There are 35 students in a class. 20 of them study English, 15 – German, 10 – French, 7 English and German, 5 – English and French, 4 German and French, 3 English, German and French. How many students do not study languages at all?

There exist formula, which is named as **inclusion-exclusion principle**, which give us the solution for that problem:

$$N(0)=N-(N1+N2+N3)+(N12+N13+N23)-N123.$$

What it means? N – Number of students. $N1$- number of student studying English, $N12$ – number of students studying English and German and so on, $N(0)$ – number of students, which don't study languages.

Why this formula is correct?



Let's represent our sets with circles. $N1$-1st circle and so on. Then the intersection of 1st and 2nd circle must be equal to $N12$ and intersection of all three circles to $N123$.

From the other side, our circles divide plane to 8 parts.
So N is equal to sum of all the parts.

$$N12=b1+c.$$

It is easy to check that after substituting $a1,\ldots$ to the formula we get f. For example, $b1$ is a part of N, $N1$, $N2$, $N12$, hence after substituting $b1$ will eliminate and so on.

Now we apply this principle to solve next problem.

Problem.

The lucky tickets. There are lots of trams in Saint Petersburg. To go in tram you must buy a ticket. Every ticket has number combined of 6 digits. For example, 026173. We call ticket to be lucky if sum of first three digits is equal to sum of last three digits. For example number 126173 is not lucky, but 126171 is lucky. A question is how many lucky tickets we have and therefore what is probability to get lucky ticket.

It is not evident question. But we will try to use main combinatorial principle to solve it. First of all we prove that.

Lemma.

Number of lucky tickets is equal to number of tickets, for which sum of digits is equal to 27.
In Russia this problem was proposed to children of 3-4 grades (Journal "Quantum"). Why? Is it so easy? It is not easy but solution is very simple. We can write one-to-one correspondence in such a way:

$$A1\ A2\ A3\ A4\ A5\ A6\ \leftarrow\rightarrow\ A1\ A2\ A3\ (9\text{-}A4)\ (9\text{-}A5)\ (9\text{-}A6)$$

Now we apply previous result to solve the equation

$$A1+A2+A3+A4+A5+A6=27$$

We know that it has $C(27+5; 5)$ solutions. Is it true answer? Not. Why? Because we have some restrictions which we have not in previous problem $Ak \le 9$. How to eliminate **superfluous** solutions? Let's use the inclusion-exclusion formula. Let's define the number with **surplus** in first place as $N1$ and so on. Then the number of numbers with no surplus will be equal to:

$$N(0)=N-(N1+N2+N3+N4+N5+N6)-(N12+N13+N14+\ldots+N56).$$

We remark that it is impossible to have surplus in more than 2 places (because then sum of numbers standing on this places must be equal to 30 which is more than 27).

How to compute $N1$? Try to use one-to-one correspondence again and again. If we subtract from surplus place 10 we get number with sum 17. Then $N1=C(17+5;5)$. The same is for $N2,\ldots,N6$. Analogous if we subtract 10 twice from place with surplus of $N12$, we get number with a sum 7. Then $N12=C(7+5;5)$. The same we can do with $N13,\ldots,N56$.

So we receive such a result:

$$N(0)=C_5^{32}-6C_5^{22}+15C_5^{12}=55252.$$

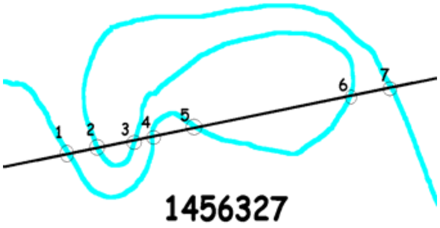
So probability to get lucky ticket is approximately equal to 5.5%.

Now you can see that we have solved difficult problem almost without computing. What we do? We try to find one-to-one correspondence with other problems and to apply basic principles.

Now we consider the problem, which was proposed by famous Russian mathematician Vladimir Arnold. This is unsolved problem for common case. Here we consider partial case of $n \le 8$.

In this case there is a way to solve it using simple combinatorial ideas and to produce recurrence relation.

Arnold problem: number of meanders



The problem can be formulated in such an evident manner.

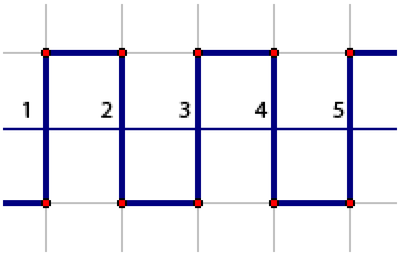
The river goes from the west to east. A straight road crosses a river. If to enumerate the intersections along the road by numbers 1, 2,…,*n*, then when moving along the river we get a permutation.

How many permutations can be constructed by varying the form of the river?

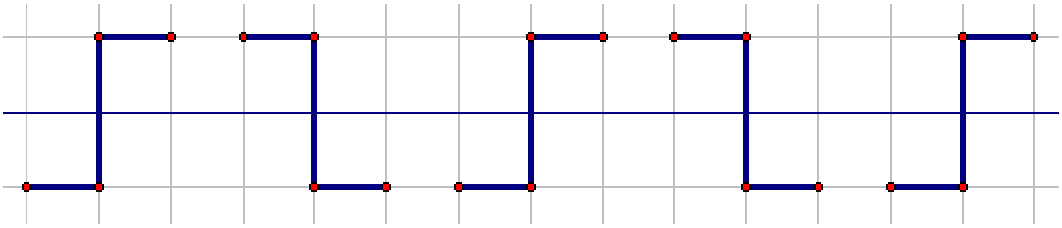
For our combinatorial reasoning we will use a piece of paper with a grid. We will draw a road as a line and a river as a curve combined as a sequence of vertical and horizontal segments.

Now we can define a wideness of meander as a minimum wide of stripe, which cover the river (we must consider all possible drawings of the river to find strip with minimal width). For example, a simple meander that is given by permutation (123…*n*) has wideness equal to 2 (see pic).

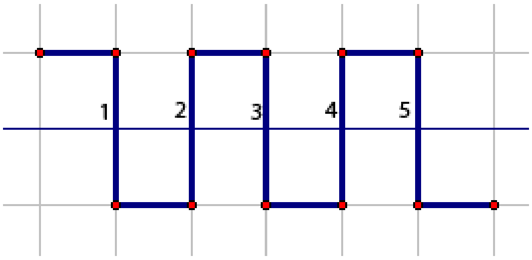
Remark. We will name the meander with *n* intersections as *n*-meander.



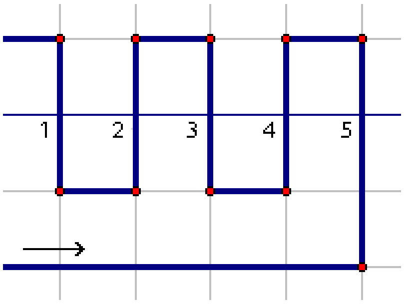
So we receive all the meanders with wideness equal to 2 because there is only one meander of such a type. This *n*-meander can be splited to a sequence of elementary 1-meanders as is shown on a pic.



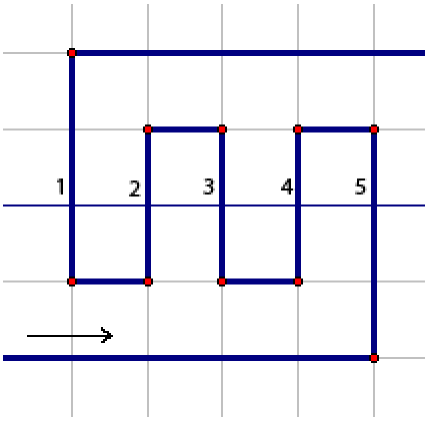
We can recollect these elementary 1-meanders to receive new river, but this new river will be described with the same permutation (123…*n*).



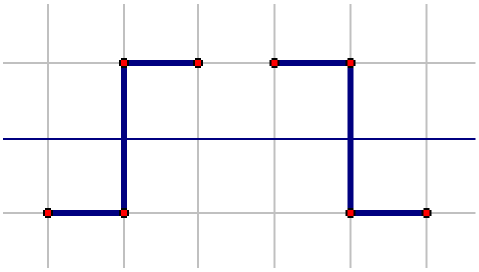
Now consider meanders with wideness equal to 3. Such a meanders could have parts with an opposite stream (see a pic.). This means the opposite order in permutation. But if we have only two parts of a river – one with direct stream and another with opposite one – the river must return to the same side of the world (from the west to west - not to the east - in our particular case). This contradicts to a given properties of a river. So we must have an odd number of river turns and there is no one meanders with wideness equal to 3.

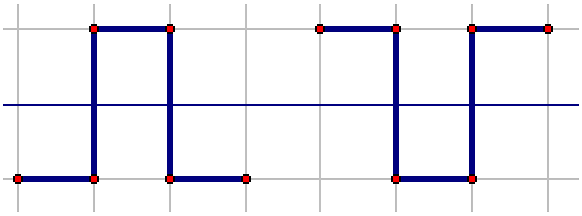


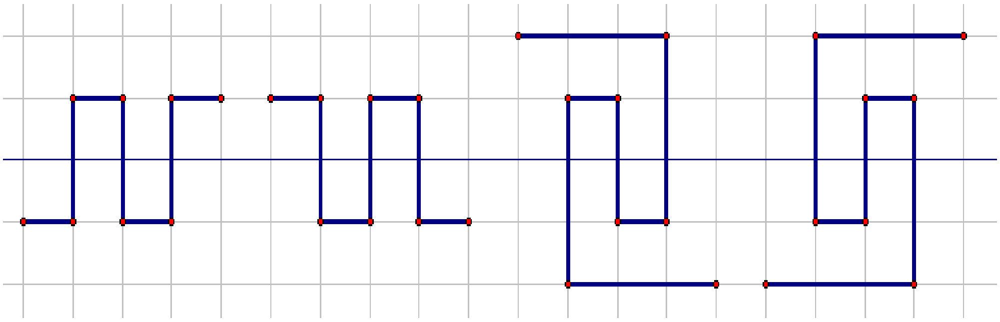
So we must go on analyzing of meanders of wideness equal to 4.



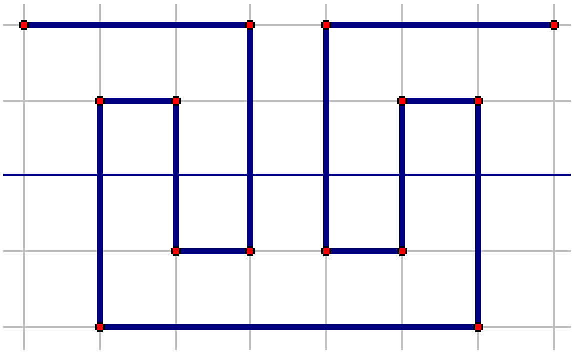
We will try to classify such a meanders using the parameter – number of intersections. We have only one 1-meander (1), one 2-meander (12) and two 3-meanders: (123), (321) (pic.). The first of 3-meander can be splited to a sequence of 1-meanders; the second 3-meander couldn’t be split to such a sequence. We will name such meanders as elementary meanders.







Note that sequence of two odd meanders gives us even meander. For example, on a pic. you can see how two 3-meanders gives a 6-meander. It is interesting that for $n<8$ every even meander can be splited to two odd meanders. Of course, it will be not true for big values of n. You can see below two 8-meanders which are elementary but they will have wideness more than 4.



So we have now two combinatorial problems:

1) to find all elementary meanders
2) to compute number of meanders which could be splited to elementary meanders?

It is easy to see that all elementary meanders with wideness less or equal to 4 can be produced from a sequence of 1-meanders by constructing at the ends of this sequence two parts with an opposite stream (pic.).

Now we are ready to solve the problem “how many 8-meandres there are with wideness no more than 4?” We can reformulate this problem as “how many ways there are to decompose 8 to an ordered sum of odd positive integers?”

Solution can be written in a form of recurrence:

Every sum must begin with an odd term less than 8: 1, 3, 5, 7.

Denote *N(n)*-number of decomposition *n* to a sum of such a type. We will get such a recurrence:

$$N(n)=N(n-7)+N(n-5)+N(n-3)+N(n-1)$$

Then

$$N(4)=N(1)+N(3)=3$$

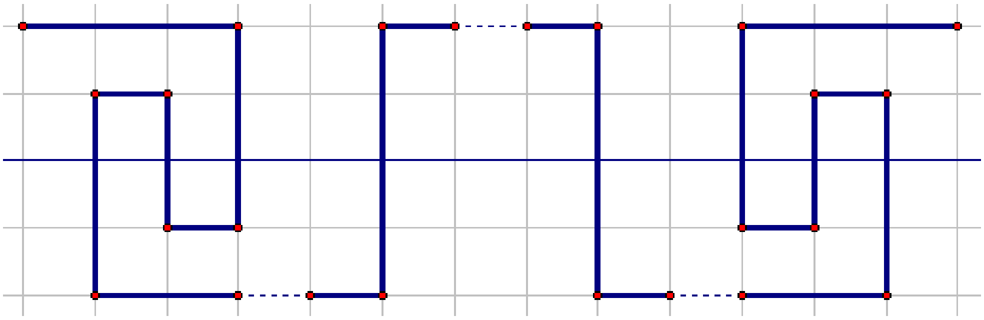
$$N(5)=N(2)+N(4)=4$$

$$N(6)=N(1)+N(3)+N(5)=7$$

$$N(7)=N(2)+N(4)+N(6)=11$$

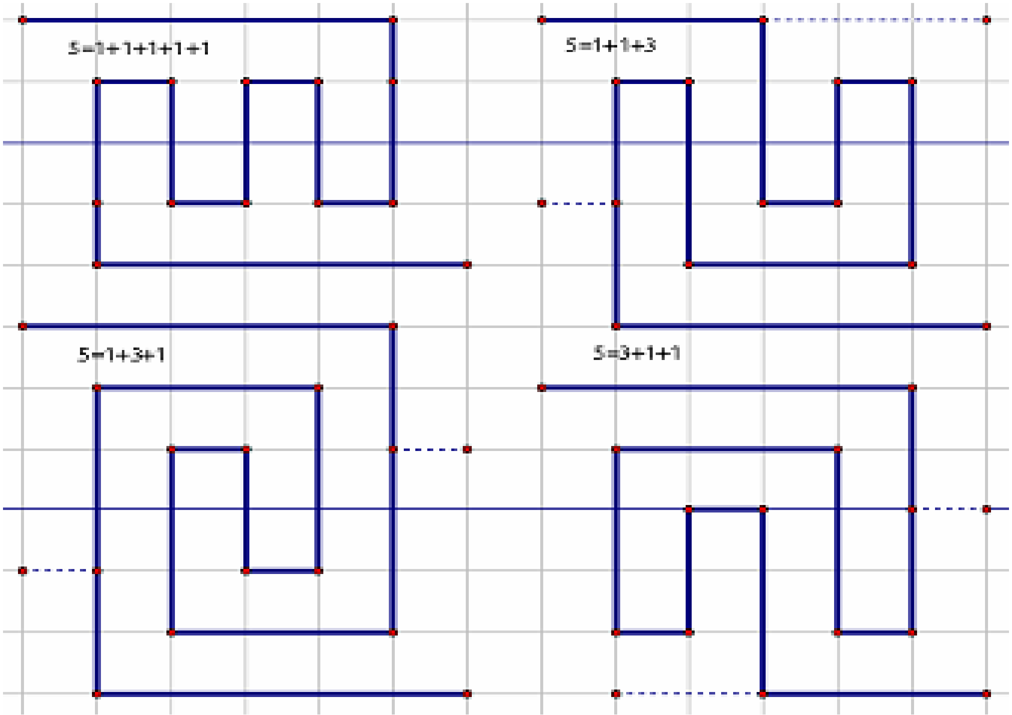
$$N(8)=N(1)+N(3)+N(5)+N(7)=18$$

It is most interesting to look how to construct a meander by decomposition of a number to an ordered sum of odd integers. For example consider the decomposition 8=3+1+1+3 and connect sequence of 3-meander, 1-meander, 1-meander and 3- meander as is shown on pic.



So we have eighteen 8-meanders with a wideness not more than 4 (one meander which is in accordance with decomposition 8=1+1+1+1+1+1+1+1 has a wideness 2).

Now we want to enlarge a wideness of meanders under consideration. Moving in such a direction we must to explore new elementary meanders. Try to find a method for generating all elementary meanders for *n*<8. We will use the same trick as for constructing elementary meanders with wideness equal to 4: we construct all not elementary meanders with wideness not more than 4 and add to the ends of them to parts with an opposite stream. The result is shown on pic.



The first of new elementary meanders is known for us (with wideness 4) but others are new.

(after that we can join all not elementary 5-meanders with elementary one to get all eight 5-meanders).

Now we must correct last recurrence relation in such a way:

$$N(n)=4N(n-7)+N(n-5)+N(n-3)+N(n-1).$$

This new recurrence gives us not more than the way to compute all not elementary 7-meanders:
$N(7)=4N(7-5)+N(7-3)+N(7-1)=4N(2)+N(4)+N(6)=4+3+(4N(6-5)+N(6-3)+N(6-1))=$
$=7+4N(1)+N(3)+N(5)=7+4+2+8=21$

Now we are ready to construct all elementary 7-meanders. To do this we will use the same trick as for 5-meanders and will get new 21 elementary 7-meanders. It is not so easy to prove that there are not other elementary 7-meanders, but it is true. Then new correction of last recurrence gives us the new one:

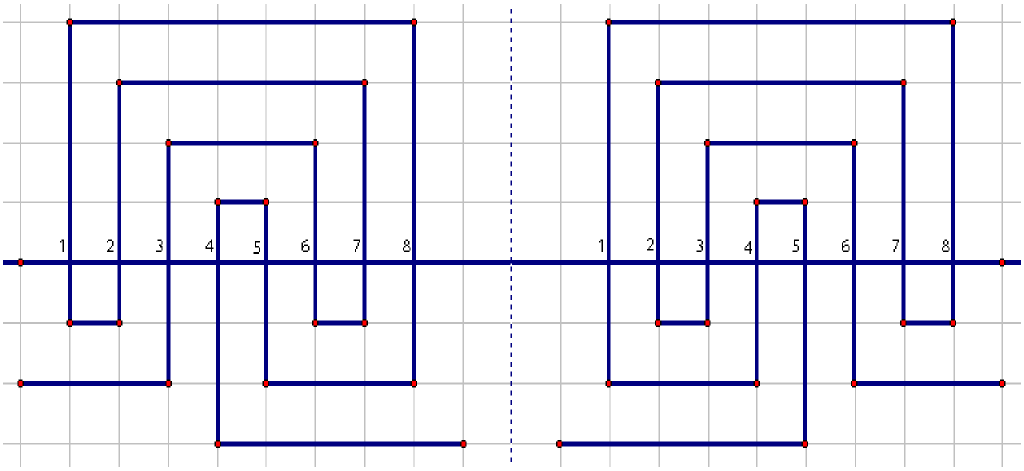
$$N(n)=21N(n-7)+4N(n-5)+N(n-3)+N(n-1)+E(n),$$

where $E(n)$ is a number of elementary n-meanders.

For n=8 we get

$$N(8)=21N(1)+4N(3)+N(5)+N(7)+E(8)=21+8+8+21+21+E(8)=79+E(8).$$

We can find two elementary 8-meanders, which are symmetrical, to each other: (36721854) and (54187236). So $E(8)$=2 and common number of 8-meanders is equal to 81.



As can be seen we have found not only the number of 8-meanders but a method for their construction.

From the other hand each new step require for number of elementary *n*-meanders. But the method discussed here gives not the algorithm for that. It is the root of problem to solve the problem in common case.