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EXACT SOLUTIONS FOR SOME UNSTEADY FLOWS OF GENERALIZED SECOND GRADE FLUIDS IN CYLINDRICAL DOMAINS

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Abstract. The velocity field and the adequate shear stress, corresponding to the unsteady flow of generalized second grade fluids due to a constantly accelerating circular cylinder, are determined by means of the Hankel and Laplace transforms. The solutions that have been obtained satisfy all imposed initial and boundary conditions and for $\beta \to 1$ reduce to the similar solutions for the second grade fluids performing the same motion.

Key words: Generalized second grade fluid, exact solutions, shear stress.

1. Introduction

Among many constitutive assumptions that have been employed to study the non-Newtonian behavior of the fluids, one class that has gained support from both the experimentalists and the theoreticians is that of Rivlin-Ericksen fluids of second grade. The Cauchy stress tensor $\mathbf{T}$ for such fluids is given by [1-7].

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \quad (1)$$

where $-p$ is the pressure, $\mathbf{I}$ is the unit tensor, μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined through

$$\mathbf{A}_1 = \mathrm{grad}\mathbf{v} + (\mathrm{grad}\mathbf{v})^T, \qquad \mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\mathrm{grad}\mathbf{v}) + (\mathrm{grad}\mathbf{v})^T\mathbf{A}_1. \quad (2)$$

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In the above relations, $\mathbf{v}$ is the velocity, d/dt denotes the material time derivative and grad the gradient operator. Since the fluid is incompressible, it can undergo only isochoric motions and hence

$$div\,\mathbf{v} = tr\mathbf{A}_1 = 0. \tag{3}$$

If this model is required to be compatible with thermodynamics, then the material moduli must meet the following restrictions

$$\mu \geq 0, \quad \alpha_1 \geq 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \tag{4}$$

The sign of the material moduli α_1 and α_2 has been the subject of much controversy. A comprehensive discussion on the restrictions given in (4), as well as a critical review on the fluids of differential type, can be found in the extensive work of Dunn and Rajagopal [8].

Generally, the constitutive equation of the generalized second grade fluids has the same form as (1), but $\mathbf{A}_2$ is defined by [9-12].

$$\mathbf{A}_2 = D_t^{\beta}\mathbf{A}_1 + \mathbf{A}_1(\text{grad}\mathbf{v}) + (\text{grad}\mathbf{v})^T\mathbf{A}_1, \tag{5}$$

where D_t^{β} is the Riemann-Liouville fractional calculus operator of order β with respect to t defined as

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta} f(\tau)\,d\tau, \quad 0 < \beta \leq 1, \tag{6}$$

where $\Gamma(\cdot)$ is the Gamma function. When $\beta = 1$, Eq. (5) may be simplified as $(2)_2$, while for $\alpha_1 = 0$ the constitutive relationship (1) describes the Rainer-Rivlin viscous fluid.

In this paper, we are interested into the motion of a generalized second grade fluid between two infinite coaxial circular cylinders, one of them sliding along their common axis with a given time-dependent velocity At. For completeness we consider the general case when both cylinders are sliding along their common axis with the velocities A_1t and A_2t. From the general case, we obtain the velocity fields and the adequate shear stresses corresponding to different special cases. The respective solutions for the motion through an infinite circular cylinder are also presented.

2. Starting Flow Between Two Concentric Cylinders

Suppose that an incompressible generalized second grade fluid at rest is situated in the annular region between two infinite straight circular cylinders of radii R_1 and $R_2(> R_1)$. At time zero, both cylinders suddenly begin to slide along their common axis $(r = 0)$ with the velocities A_1t and A_2t. Owing to the shear, the fluid between cylinders is gradually moved, its velocity being of the form

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)\,\mathbf{e}_z, \tag{7}$$

where $\mathbf{e}_z$ is the unit vector along z-axis. For such flows the constraint of incompressibility is automatically satisfied.

Introducing (7) into the constitutive equation, we find that

$$\tau(r,t) = (\mu + \alpha_1 D_t^{\beta})\frac{\partial v(r,t)}{\partial r}, \tag{8}$$

where $\tau(r,t) = S_{rz}(r,t)$ is the shear stress, which is different of zero. In the absence of body forces and a pressure gradient in the axial direction, the balance of the linear momentum leads to the relevant equation

$$\rho\frac{\partial v(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\tau(r,t). \tag{9}$$

Eliminating $\tau(r,t)$ between Eqs. (8) and (9) we get the governing equation

$$\frac{\partial v(r,t)}{\partial t} = (\nu + \alpha D_t^{\beta})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t); \quad r \in (R_1, R_2), \;\; t > 0, \tag{10}$$

where $\alpha = \alpha_1/\rho$ and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid (ρ being its constant density).

The appropriate initial and boundary conditions are

$$v(r,0) = 0, \; r \in (R_1, R_2); \; v(R_1,t) = A_1 t, \;\; v(R_2,t) = A_2 t \quad \text{for} \quad t > 0. \tag{11}$$

2.1. **Calculation of the Velocity Field.** Applying the Laplace transform to Eqs. (10) and (11) and using the Laplace transform formula for sequential fractional derivatives [13], we obtain the ordinary differential equation

$$\frac{\partial^2 \overline{v}(r,q)}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v}(r,q)}{\partial r} - \frac{q}{\alpha q^{\beta} + \nu}\overline{v}(r,q) = 0; \;\; r \in (R_1, R_2), \tag{12}$$

where the image function $\overline{v}(r,q)$ of $v(r,t)$ has to satisfy the conditions

$$\overline{v}(R_1,q) = \frac{A_1}{q^2}, \qquad \overline{v}(R_2,q) = \frac{A_2}{q^2}. \tag{13}$$

In the following, let us denote by

$$\overline{v}_n(q) = \int_{R_1}^{R_2} r\overline{v}(r,q)B_0(r\,r_n)dr; \quad n = 1,2,3,\cdots, \tag{14}$$

the finite Hankel transforms of $\overline{v}(r,q)$, where r_n are the positive roots of the transcendental equation $B_0(R_1\,r) = 0$ and

$$B_0(r\,r_n) = J_0(r\,r_n)Y_0(R_2\,r_n) - J_0(R_2\,r_n)Y_0(r\,r_n). \tag{15}$$

In the above relation, $J_0(\cdot)$ and $Y_0(\cdot)$ are Bessel functions of order zero of the first and second kind. Applying the finite Hankel transform to Eq. (12) and taking into account the conditions (13), we find that [14].

$$\frac{2[A_2 J_0(R_1\, r_n) - A_1 J_0(R_2\, r_n)]}{\pi q^2 J_0(R_1\, r_n)} - r_n^2 \overline{v}_n(q) - \frac{q}{\alpha q^\beta + \nu}\overline{v}_n(q) = 0, \tag{16}$$

or equivalently,

$$\overline{v}_n(q) = \frac{2[A_2 J_0(R_1\, r_n) - A_1 J_0(R_2\, r_n)]}{\pi J_0(R_1\, r_n)} \frac{\alpha q^\beta + \nu}{q^2[\alpha r_n^2 q^\beta + q + \nu r_n^2]}. \tag{17}$$

In order to determine $\overline{v}(r,q)$, we firstly write $\overline{v}_n(q)$ under the suitable form

$$\overline{v}_n(q) = \frac{2}{\pi r_n^2}\frac{A_2 J_0(R_1\, r_n) - A_1 J_0(R_2\, r_n)}{J_0(R_1\, r_n)}\frac{1}{q^2} -$$

$$-\frac{2}{\pi r_n^2}\frac{A_2 J_0(R_1\, r_n) - A_1 J_0(R_2\, r_n)}{J_0(R_1\, r_n)}\frac{1}{q[\alpha r_n^2 q^\beta + q + \nu r_n^2]} \tag{18}$$

and use the inverse Hankel transform formula [14].

$$\overline{v}(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_0^2(R_1\, r_n)}{J_0^2(R_1\, r_n) - J_0^2(R_2\, r_n)}\overline{v}_n(q) B_0(r\, r_n). \tag{19}$$

Furthermore, in order to avoid the burdensome calculations of residues and contour integrals, we apply the discrete inversion Laplace transform method [11, 12], writing

$$\frac{1}{q[\alpha r_n^2 q^\beta + q + \nu r_n^2]} = \frac{1}{q^{\beta+1}[\nu r_n^2 q^{-\beta} + (q^{1-\beta} + \alpha r_n^2)]} =$$

$$= \sum_{k=0}^{\infty}\frac{(-\nu r_n^2)^k q^{-\beta k - \beta - 1}}{(q^{1-\beta} + \alpha r_n^2)^{k+1}} \tag{20}$$

and use Eq. (A2), where [15]

$$G_{a,\,b,\,c}(d_n, t) = \sum_{j=0}^{\infty}\frac{(c)_j (d_n)^j t^{(j+c)a-b-1}}{\Gamma(j+1)\Gamma[(j+c)a-b]}; \quad Re(ac-b) > 0, \; Re(q) > 0, \tag{21}$$

$a = 1-\beta$, $b = -\beta k - \beta - 1$, $c = k+1$, $d_n = -\alpha r_n^2$, $|d_n/q^a| < 1$ and $(c)_j$ is the Pochhammer polynomial [15].

Finally, Eqs. (18)-(20), (A1) and (21) imply

$$v(r,t) = \frac{A_1 \ln(R_2/r) + A_2 \ln(r/R_1)}{\ln(R_2/R_1)} t -$$

$$-\pi\sum_{n=1}^{\infty}\frac{J_0(R_1r_n)[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)]}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}B_0(rr_n)\times$$
$$\times\sum_{k=0}^{\infty}(-\nu r_n^2)^kG_{a,\,b,\,c}(-\alpha r_n^2,t), \tag{22}$$

or equivalently,

$$v(r,t)=\frac{A_1\ln(R_2/r)+A_2\ln(r/R_1)}{\ln(R_2/R_1)}t-$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_0(R_1r_n)[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)]}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}B_0(rr_n)\times$$
$$\times\sum_{j,\,k=0}^{\infty}\frac{(-\alpha r_n^2)^j(-\nu r_n^2)^k(k+1)_j\,t^{k+1+(1-\beta)j}}{\Gamma(j+1)\Gamma[k+2+(1-\beta)j]}. \tag{23}$$

2.2. **Calculation of Shear Stress.** Applying the Laplace transform to Eq. (8), we find that

$$\overline{\tau}(r,q)=(\mu+\alpha_1q^\beta)\frac{\partial\overline{v}(r,q)}{\partial r}, \tag{24}$$

where

$$\frac{\partial\overline{v}(r,q)}{\partial r}=\frac{A_2-A_1}{r\ln(R_2/R_1)}\frac{1}{q^2}+\pi\sum_{n=1}^{\infty}\frac{r_nJ_0(R_1r_n)[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)]}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times$$
$$\times B_{01}(rr_n)\sum_{j,\,k=0}^{\infty}\frac{(-\alpha r_n^2)^j(-\nu r_n^2)^k(k+1)_j}{\Gamma(j+1)}\frac{1}{q^{k+2+(1-\beta)j}}, \tag{25}$$

has been obtained from (23) and

$$B_{01}(rr_n)=J_1(rr_n)Y_0(R_2r_n)-J_0(R_2r_n)Y_1(rr_n).$$

Introducing (25) into (24) and applying again the discrete inversion Laplace transform method to the obtained result, we find for the shear stress the expression

$$\tau(r,t)=\frac{\mu(A_2-A_1)}{r\ln(R_2/R_1)}t+\frac{\alpha_1(A_2-A_1)}{r\ln(R_2/R_1)}\frac{t^{1-\beta}}{\Gamma(2-\beta)}+$$
$$+\pi\sum_{n=1}^{\infty}\frac{r_nJ_0(R_1r_n)[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)]}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times$$
$$\times B_{01}(rr_n)\sum_{j,\,k=0}^{\infty}\frac{(-\alpha r_n^2)^j(-\nu r_n^2)^k(k+1)_j}{\Gamma(j+1)}\times$$

$$\times\left[\frac{\mu t^{k+1+(1-\beta)j}}{\Gamma[k+2+(1-\beta)j)]}+\frac{\alpha_1 t^{k+1+(1-\beta)j-\beta}}{\Gamma[k+2+(1-\beta)j-\beta]}\right]. \tag{26}$$

3. Starting Flow Through a Circular Cylinder

Let us now assume that our fluid is at rest in an infinite circular cylinder of radius R. At time $t=0^+$, the cylinder is subject to a translation along its axis with a time dependent velocity At. Due to the shear the fluid is gradually moved, its velocity and the governing equation being of the same forms as (7) and (10), respectively. The corresponding initial and boundary conditions are

$$v(r,0)=0, \quad r\in[0,R); \qquad v(R,t)=At, \quad t>0. \tag{27}$$

Applying again the Laplace transform to Eq. (10) and having in mind the initial and boundary conditions (27), we find for $\overline{v}(r,q)$ the same ordinary differential equation (12), with the condition

$$\overline{v}(R,q)=A/q^2. \tag{28}$$

Now, multiplying Eq. (12) by $rJ_0(rr_n)$ where r_n are the positive roots of the transcendental equation $J_0(Rr)=0$ and integrating with respect to r from 0 to R, we find for the new Hankel transforms

$$\overline{v}_n(q)=\int_0^R r\overline{v}(r,q)J_0(rr_n)dr; \quad n=1,2,3,\cdots, \tag{29}$$

of $\overline{v}(r,q)$ the expression (see also Eq. (28) and (A3))

$$\overline{v}_n(q)=ARr_nJ_1(Rr_n)\frac{\nu+\alpha q^\beta}{q^2(q+\nu r_n^2+\alpha r_n^2 q^\beta)}, \tag{30}$$

or equivalently,

$$\overline{v}_n(q)=\frac{AR}{r_n}J_1(Rr_n)\left[\frac{1}{q^2}-\frac{1}{q(q+\nu r_n^2+\alpha r_n^2 q^\beta)}\right]. \tag{31}$$

Applying the inverse Hankel transform [14] to Eq. (31) and using (A3), it results that

$$\overline{v}(r,q)=\frac{A}{q^2}-\frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(Rr_n)}\,\frac{1}{q(q+\nu r_n^2+\alpha r_n^2 q^\beta)}. \tag{32}$$

Finally, following the same way as before, we find for $v(r,t)$ the expression

$$v(r,t)=At-\frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(Rr_n)}\sum_{k=0}^{\infty}(-\nu r_n^2)^k G_{a,b,c}(-\alpha\, r_n^2,t)=$$

$$= At - \frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(Rr_n)}\sum_{j,\,k=0}^{\infty}\frac{(-\alpha\, r_n^2)^j(-\nu r_n^2)^k(k+1)_j}{\Gamma(j+1)}\times$$
$$\times\frac{t^{k+1+(1-\beta)j}}{\Gamma[k+2+(1-\beta)j]}. \tag{33}$$

Introducing (33) into (24) we find

$$\overline{\tau}(r,q) = \frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(Rr_n)}\sum_{j,\,k=0}^{\infty}\frac{(-\alpha\, r_n^2)^j(-\nu r_n^2)^k(k+1)_j}{\Gamma(j+1)}\times$$
$$\times\left[\frac{\mu}{q^{k+2+(1-\beta)j}}+\frac{\alpha_1}{q^{k+2+(1-\beta)j-\beta}}\right] \tag{34}$$

and from here the shear stress

$$\tau(r,t) = \frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(Rr_n)}\sum_{j,\,k=0}^{\infty}\frac{(-\alpha\, r_n^2)^j(-\nu r_n^2)^k(k+1)_j}{\Gamma(j+1)}\times$$
$$\times\left[\frac{\mu t^{k+1+(1-\beta)j}}{\Gamma[k+2+(1-\beta)j]}+\frac{\alpha_1 t^{k+1+(1-\beta)j-\beta}}{\Gamma[k+2+(1-\beta)j-\beta]}\right]. \tag{35}$$

4. The Special Case: $\beta \to 1$

Making $\beta \to 1$ into Eqs. (23), (26), (33) and (35) we obtain the similar solutions for a second grade fluid performing the same motion. The last two equalities become

$$v(r,t) = At - \frac{2At}{R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(Rr_n)}\sum_{j,\,k=0}^{\infty}\frac{(-\alpha\, r_n^2)^j(-\nu\, t\, r_n^2)^k(k+1)_j}{\Gamma(j+1)\Gamma(k+2)} \tag{36}$$

and

$$\tau(r,t) = \frac{2\rho At}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(Rr_n)}\sum_{j,\,k=0}^{\infty}\frac{(-\alpha\, r_n^2)^j(-\nu\, t\, r_n^2)^k(k+1)_j}{\Gamma(j+1)\Gamma(k+1)}\left(\frac{\nu}{k+1}+\frac{\alpha}{t}\right). \tag{37}$$

On the other hand, Eq. (32) for $\beta \to 1$ can be written in the suitable form

$$\overline{v}(r,q) = \frac{A}{q^2} - \frac{2A}{\nu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^3J_1(Rr_n)}\left[\frac{1}{q}-\frac{\frac{1+\alpha r_n^2}{\nu r_n^2}}{1+\frac{1+\alpha r_n^2}{\nu r_n^2}q}\right], \tag{38}$$

from which it immediately results the solution

$$v(r,t) = At - \frac{2A}{\nu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^3J_1(Rr_n)}\left[1-\exp\left(-\frac{\nu r_n^2}{1+\alpha r_n^2}t\right)\right], \tag{39}$$

obtained in [17] by a different method. From Eqs. (36) and (39) we get the identity

$$\frac{1}{\nu\, r_n^2}\left[1-\exp\left(-\frac{\nu\, r_n^2}{1+\alpha r_n^2}t\right)\right]=\sum_{j,\,k=0}^{\infty}\frac{(-\alpha\, r_n^2)^j(-\nu\, r_n^2)^k(k+1)_j}{\Gamma(j+1)\Gamma(k+2)}\; t^{k+1}, \quad (40)$$

that has been numerically proved.

Introducing (38) into (24) and making all calculi, we again attain to the known result

$$\tau(r,t)=\frac{2\rho A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^2 J_1(Rr_n)}\left[1-\frac{1}{1+\ \alpha r_n^2}\exp\left(-\frac{\nu\, r_n^2}{1+\alpha r_n^2}t\right)\right], \quad (41)$$

that together with (37) implies the identity

$$1-\frac{1}{1+\ \alpha r_n^2}\exp\left(-\frac{\nu\, r_n^2}{1+\alpha r_n^2}t\right)=\sum_{j,\,k=0}^{\infty}\frac{(-\alpha\, r_n^2)^j(-\nu\, r_n^2)^k(k+1)_j}{\Gamma(j+1)\Gamma(k+1)}\times$$

$$\times\left(\frac{\nu r_n^2}{k+1}+\frac{\alpha r_n^2}{t}\right)\, t^{k+1}. \quad (42)$$

which, of course, has been also proved numerically.

5. Conclusion

Our purpose in this paper was to establish exact solutions for the velocity field and shear stress corresponding to the flow of a generalized second grade fluid due to an infinite circular cylinder subject to a translation along its axis with a velocity of constant acceleration A. However, for completeness, we have also considered the case of the flow between two coaxial circular cylinders, both cylinders have been assumed to slide along their common axis with velocities of constant accelerations A_1 and A_2. Making $A_1=0$ and $A_2=A$ into (23), for instance, we obtain the velocity field

$$v(r,t)=At\frac{\ln(r/R_1)}{\ln(R_2/R_1)}-\pi At\sum_{n=1}^{\infty}\frac{J_0^2(R_1r_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\,B_0(rr_n)\times$$

$$\times\sum_{j,\,k=0}^{\infty}\frac{(-\alpha\, r_n^2)^j(-\nu\, r_n^2)^k(k+1)_j}{\Gamma(j+1)\Gamma[k+2+(1-\beta)j]}\; t^{k+(1-\beta)j}, \quad (43)$$

corresponding to the flow between cylinders, the inner cylinder being at rest.

The solutions that have been obtained, presented under integral and series forms in terms of the generalized functions $G_{a,\,b,\,c}(d,t)$, satisfy all imposed initial and boundary conditions and for $\beta\to 1$ reduce to the similar solutions for second grade fluids. Finally, the solutions for the flow through an infinite circular cylinder have been also established and some known results have been recovered as special cases of our general solutions.

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Appendix

Some results used in the text:

The finite Hankel transform of the function

$$a(r) = \frac{A_2 - A_1}{\ln(R_2/R_1)} \ln r + \frac{A_1 \ln R_2 - A_2 \ln R_1}{\ln(R_2/R_1)},$$

satisfying $a(R_1) = A_1$ and $a(R_2) = A_2$ is

$$a_n = \int_{R_1}^{R_2} r a(r)\, B_0(r r_n) dr = \frac{2}{\pi r_n^2} \frac{A_2 J_0(R_1 r_n) - A_1 J_0(R_2 r_n)}{J_0(R_1 r_n)}. \tag{A1}$$

In order to prove (A1), we integrate by parts and use the next identities:

$$\int J_1(u) du = -J_0(u), \qquad J_1(R_1 r_n) Y_0(R_1 r_n) - J_0(R_1 r_n) Y_1(R_1 r_n) = \frac{2}{\pi R_1 r_n}$$

and

$$J_1(R_2 r_n) Y_0(R_2 r_n) - J_0(R_2 r_n) Y_1(R_2 r_n) = \frac{2}{\pi R_2 r_n} \quad \text{if} \quad B_0(R_1 r_n) = 0.$$

$$L^{-1}\left\{\frac{q^b}{(q^a - d)^c}\right\} = G_{a,b,c}(d,t); \;\; Re(ac - b) > 0, \;\; Re(q) > 0, \;\; \left|\frac{d}{q^a}\right| < 1. \tag{A2}$$

$$\int_0^R r J_0(r r_n) dr = \frac{R}{r_n} J_1(R r_n). \tag{A3}$$