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ON THE CONNECTED DETOUR NUMBER OF A GRAPH

A. P. SANTHAKUMARAN[1], S. ATHISAYANATHAN[2]

ABSTRACT. For two vertices u and v in a graph $G = (V, E)$, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a u–v *detour*. A set $S \subseteq V$ is called a *detour set* of G if every vertex in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of its detour sets and any detour set of order $dn(G)$ is a *detour basis* of G. A set $S \subseteq V$ is called a *connected detour set* of G if S is detour set of G and the subgraph $G[S]$ induced by S is connected. The *connected detour number* $cdn(G)$ of G is the minimum order of its connected detour sets and any connected detour set of order $cdn(G)$ is called a *connected detour basis* of G. Graphs G with detour diameter $D \leq 4$ are characterized when $cdn(G) = p$, $cdn(G) = p-1$, $cdn(G) = p-2$ or $cdn(G) = 2$. A subset T of a connected detour basis S of G is a *forcing subset* for S if S is the unique connected detour basis containing T. The *forcing connected detour number* $fcdn(S)$ of S is the minimum cardinality of a forcing subset for S. The *forcing connected detour number* $fcdn(G)$ of G is $\min\{fcdn(S)\}$, where the minimum is taken over all connected detour bases S in G. The forcing connected detour numbers of certain classes of graphs are determined. It is also shown that for each pair a, b of integers with $0 \leq a < b$ and $b \geq 3$, there is a connected graph G with $fcdn(G) = a$ and $cdn(G) = b$.

Key words: detour, connected detour set, connected detour basis, connected detour number, forcing connected detour number.
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1. INTRODUCTION

By a *graph* $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively.

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We consider connected graphs with at least two vertices. For basic definitions and terminologies, we refer to [1, 4].

For vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. It is known that the detour distance is a metric on the vertex set V. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D G$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D G$ of G is the maximum detour eccentricity among the vertices of G. These concepts were studied by Chartrand et al. [2].

A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. A vertex v that belongs to every detour basis of G is a *detour vertex* in G. If G has a unique detour basis S, then every vertex in S is a detour vertex in G. These concepts were studied by Chartrand et al. [3].

The concepts of connected detour number and upper connected detour number were introduced and studied by Santhakumaran and Athisayanathan in [5]. A set $S \subseteq V$ is called a *connected detour set* of G if S is a detour set of G and the subgraph $G[S]$ induced by S is connected. The *connected detour number* $cdn(G)$ of G is the minimum order of its connected detour sets and any connected detour set of order $cdn(G)$ is called a *connected detour basis* of G. A vertex v in a graph G is a *connected detour vertex* if v belongs to every connected detour basis of G. If G has a unique connected detour basis S, then every vertex in S is a connected detour vertex of G. These concepts have interesting applications in the *Channel Assignment Problem* in radio technologies.

For the graph G given in Figure 1.1, the sets $S_1 = \{v_1, v_3\}$, $S_2 = \{v_1, v_5\}$ and $S_3 = \{v_1, v_4\}$ are the three detour bases of G so that $dn(G) = 2$. It is clear that no two element subset of V is a connected detour set of G. However the set $S_4 = \{v_1, v_2, v_3\}$ is a connected detour basis of G so that $cdn(G) = 3$. Also the set $S_5 = \{v_1, v_2, v_5\}$ is another connected detour basis of G. Thus there can be more than one connected detour basis for a graph G.

The following theorems are used in the sequel.

Theorem 1.1. [5] *For any graph G of order $p \geq 2$, $2 \leq dn(G) \leq cdn(G) \leq p$.*

Theorem 1.2. [5] *Let G be a connected graph with cut-vertices and S a connected detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

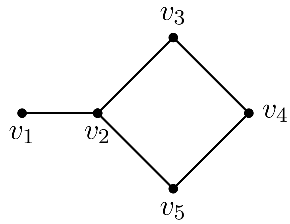


FIGURE 1.1. G

Theorem 1.3. [5] *All the end-vertices and the cut-vertices of a connected graph G belong to every connected detour set of G.*

Theorem 1.4. [5] *If G is a graph of order $p \geq 2$ such that every vertex v of G is either an end-vertex or a cut-vertex, then $cdn(G) = p$.*

Theorem 1.5. [5] *If T is a tree of order $p \geq 2$, then $cdn(T) = p$.*

Theorem 1.6. [5] *Let $G = (K_{n_1} \cup K_{n_2} \cup ... \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geq 4$ such that $r \geq 1$, each $n_i \geq 2$ and $n_1 + n_2 + \ldots + n_r + k = p - 1$. Then $cdn(G) = r + k + 1$.*

Theorem 1.7. [5] *Let G be a Hamilton graph of order $p \geq 3$. Then $cdn(G) = 2$.*

Theorem 1.8. [5] *Let G be the complete graph K_p $(p \geq 2)$ or the cycle C_p or the complete bipartite graph $K_{m,n}$ $(m, n \geq 2)$. Then a set S of vertices is a connected detour basis if and only if S consists of two adjacent vertices of G.*

Theorem 1.9. [5] *If G is the complete graph K_p $(p \geq 2)$ or the cycle C_p or the complete bipartite graph $K_{m,n}$ $(m, n \geq 2)$, then $cdn(G) = 2$.*

Chartrand et al. [3] proved the following theorem, which gives an upper bound for the detour number of a graph in terms of its order and detour diameter D.

Theorem 1.10. [3] *If G is a nontrivial connected graph of order p and detour diameter D, then $dn(G) \leq p - D + 1$.*

Theorem 1.10 does not hold in the case of connected detour number $cdn(G)$ of a graph G. Santhakumaran and Athisayanathan [5] showed the existence of graphs G for which $cdn(G) = p - D + 1$, $cdn(G) > p - D + 1$ and $cdn(G) < p - D + 1$. For the graph G in Figure 1.2(a), $p = 8$, $D = 4$ and by Theorem 1.6, $cdn(G) = 5$ so that $cdn(G) = p - D + 1$. It also follows similarly from Theorem 1.6 that for the graphs G in Figure 1.2(b) and Figure 1.2(c), $cdn(G) > p - D + 1$ and $cdn(G) < p - D + 1$ respectively.

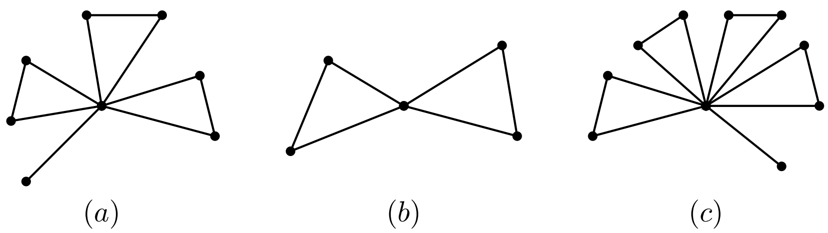


Figure 1.2. G

Some interesting results regarding the relation of the connected detour number of a graph G with regard to its order and detour diameter, and also realization results when the detour diameter of G is at least 4 are proved in [5].

Throughout this paper G denotes a connected graph with at least two vertices.

2. Graphs with Detour Diameter $D \leq 4$ and Connected Detour Numbers p, $p-1$, $p-2$ or 2

In view of Theorem 1.1, we proceed to characterize graphs G with detour diameter $D \leq 4$ for which $cdn(G) = p$ or $cdn(G) = p-1$ or $cdn(G) = p-2$ or $cdn(G) = 2$.

First we characterize graphs G with detour diameter $D \leq 4$ for which $cdn(G) = p$. For this purpose we introduce the graph G given in Figure 2.1

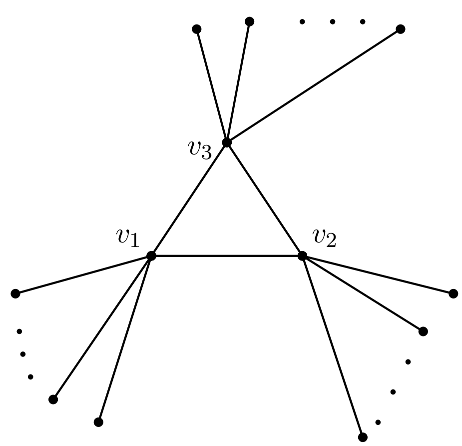


Figure 2.1. G

Theorem 2.1. *Let G be a connected graph of order $p \geq 2$ with detour diameter $D \leq 4$. Then $cdn(G) = p$ if and only if G is a tree with $D \leq 4$ or the graph G given in* Figure 2.1.

Proof. If G is a tree, then by Theorem 1.5, $cdn(G) = p$. If G is the graph as given in Figure 2.1, then by Theorem 1.4, $cdn(G) = p$.

For the converse, let G be a connected graph with detour diameter $D \leq 4$ and $cdn(G) = p$. If G is a tree, then by Theorem 1.5, $cdn(G) = p$ and so it is enough to prove the result when G is not a tree. Assume that G is not a tree. **Suppose $\mathbf{D = 2}$**. Let $c(G)$ denote the length of a longest cycle in G. Since G is connected and $D = 2$, it follows that $c(G) = 3$ and G has exactly three vertices. Hence $G = K_3$ and by Theorem 1.9, $cdn(G) = 2 = p - 1$. Thus, in this case there are no graphs satisfying the requirements of the theorem.
Suppose $\mathbf{D = 3}$. Let $c(G)$ denote the length of a longest cycle in G. Since G is connected and $D = 3$, it follows that $c(G) \leq 4$. We consider two cases.
Case 1: Let $c(G) = 4$. Then, since G is connected and $D = 3$, it is clear that G has exactly four vertices. Then the graph G reduces to C_4, $K_4 - e$ or K_4. By Theorem 1.9, $cdn(C_4) = cdn(K_4) = 2 = p - 2$. Also, if $G = K_4 - e$, then $cdn(K_4 - e) = 2 = p - 2$. Thus in this case there are no graphs satisfying the requirements of the theorem.
Case 2: Let $c(G) = 3$. If G contains two or more triangles, then $c(G) = 4$ or $D \geq 4$, which is a contradiction. Hence G contains a unique triangle $C_3 : v_1, v_2, v_3, v_1$. Now, if there are two or more vertices of C_3 having degree 3 or more, then $D \geq 4$, which is contradiction. Thus exactly one vertex in C_3 has degree 3 or more. Since $D = 3$ and G is connected, it follows that $G = K_{1,p-1} + e$ and so by Theorem 1.6, $cdn(K_{1,p-1} + e) = 1 + p - 3 + 1 = p - 1$. Thus, in this case also there are no graphs satisfying the requirements of the theorem.
Suppose $\mathbf{D = 4}$. Let $c(G)$ denote the length of a longest cycle in G. Since G is connected and $D = 4$, it follows that $c(G) \leq 5$. We consider three cases.
Case 1: Let $c(G) = 5$. Then since $D = 4$, it is clear that G has exactly five vertices and each of the graph is Hamiltonian and so by Theorem 1.7, $cdn(G_i) = 2 = p - 3$. Thus in this case there are no graphs satisfying the requirements of the theorem.
Case 2: Let $c(G) = 4$. Suppose that G contains K_4 as an induced subgraph. Since $p \geq 5$, $D = 4$ and $c(G) = 4$, every vertex not on K_4 is pendent and adjacent to exactly one vertex of K_4. Thus the graph reduces to the graph G given in Figure 2.2. Now, by Theorem 1.6, $cdn(G) = 1 + (p - 4) + 1 = p - 2$. So, in this case also there are no graphs satisfying the requirements of the theorem.

Now, suppose that G does not contain K_4 as an induced subgraph. We claim that G contains exactly one 4-cycle C_4. Suppose that G contains two or more 4-cycles. If two 4-cycles in G have no edges in common, then it is

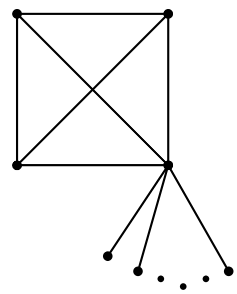


FIGURE 2.2. G

clear that $D \geq 5$, which is a contradiction. If two 4-cycles in G have exactly one edge in common, then G must contain the graphs given in Figure 2.3 as subgraphs or induced subgraphs. In any case $D \geq 5$ or $c(G) \geq 5$, which is a contradiction.

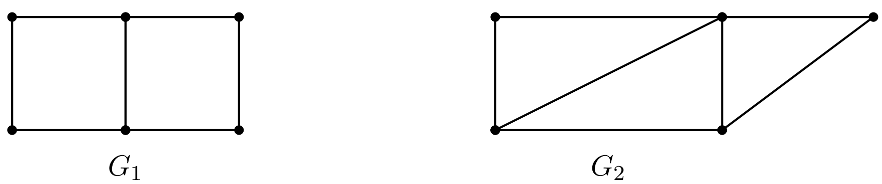


FIGURE 2.3. G

If two 4-cycles in G have exactly two edges in common, then G must contain the graphs given in Figure 2.4 as subgraphs. It is easily verified that all other subgraphs having two edges in common will have cycles of length ≥ 5 so that $D \geq 5$, which is a contradiction.

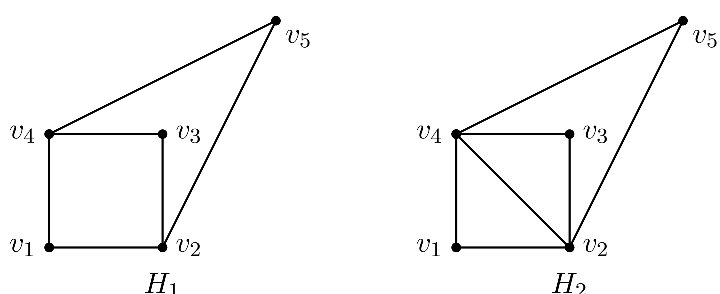


FIGURE 2.4.

Now, if $G = H_1$, then $\{v_1, v_2\}$ is a connected detour basis and so $cdn(G) = 2 = p - 3$. Hence $G \neq H_1$. Next, assume that G contains H_1 as a proper subgraph. Then there is a vertex x such that $x \notin V(H_1)$ and x is adjacent to at least one vertex of H_1. If x is adjacent to v_1, then there is a path $x, v_1, v_2, v_3, v_4, v_5$ of length 5 so that $D \geq 5$, which is a contradiction. Hence x cannot be adjacent to v_1. Similarly x cannot be adjacent to v_3 and v_5. Thus x is adjacent to v_2 or v_4 or both. If x is adjacent only to v_2, then x must be a pendant vertex of G, for otherwise, there is a path of length 5 so that $D \geq 5$, which is a contradiction. Thus in this case, the graph G reduces to the one given in Figure 2.5.

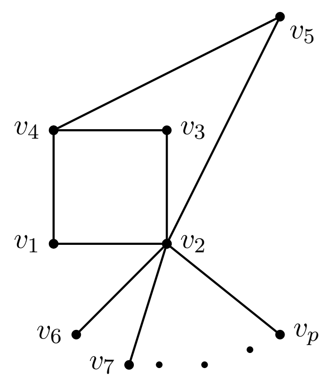


FIGURE 2.5. G

But, for this graph G, it follows from Theorem 1.3 that the set $S = \{v_1, v_2, v_6, v_7, \ldots, v_p\}$ is a connected detour basis so that $cdn(G) = p-3$. So, in this case also there are no graphs satisfying the requirements of the theorem. If x is adjacent only to v_4, then we get a graph G isomorphic to the one given in Figure 2.5 and hence in this case also there are no graphs satisfying the requirements of the theorem. If x is adjacent to both v_2 and v_4, then the graph reduces to the one given in Figure 2.6.

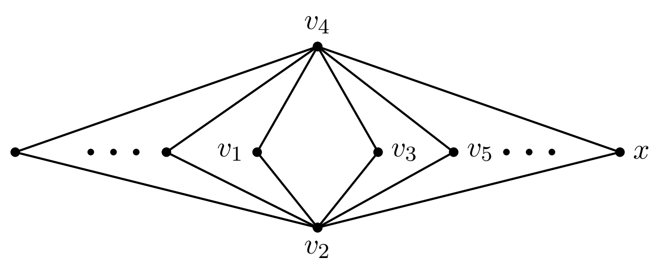


FIGURE 2.6. G

However for this graph, $\{v_1, v_2\}$ is a connected detour basis so that $cdn(G) = 2$ and hence $cdn(G) \leq p-4$, which is a contradiction. Thus a vertex not in H_1 cannot be adjacent to both v_2 and v_4.

Next, if a vertex x not on H_1 is adjacent only to v_2 and a vertex y not on H_1 is adjacent only to v_4, then x and y must be pendant vertices of G, for otherwise, we get either a path or a cycle of length ≥ 5 so that $D \geq 5$, which is a contradiction. Thus in this case, the graph reduces to the one given in Figure 2.7.

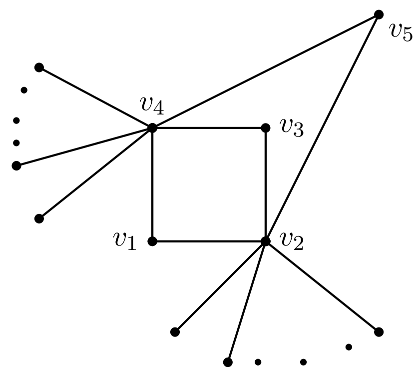


FIGURE 2.7. G

But, for this graph G, it follows from Theorem 1.3 that the set of all end-vertices together with the cut-vertices v_2 and v_4 and the vertex v_1 is a connected detour basis so that $cdn(G) = p - 2$. So, in this case also there are no graphs satisfying the requirements of the theorem. Thus we conclude that in this case there are no graphs G with H_1 as proper subgraph.

Next, if $G = H_2$, then $\{v_2, v_4\}$ is a connected detour basis and so $cdn(G) = 2 = p - 3$. Hence $G \neq H_2$. Now, assume that G contains H_2 as a proper subgraph. Then there is a vertex x such that $x \notin V(H_2)$ and x is adjacent to at least one vertex of H_2. If x is adjacent to v_1, we get a path $x, v_1, v_2, v_3, v_4, v_5$ of length 5 so that $D \geq 5$, which is a contradiction. Hence x cannot be adjacent to v_1. Similarly x cannot be adjacent to v_3 and v_5. Thus x is adjacent to v_2 or v_4 or both. If x is adjacent only to v_2, then x must be a pendant vertex of G, for otherwise, we get a path of length 5 so that $D \geq 5$, which is a contradiction. Thus in this case, the graph G reduces to the one given in Figure 2.8. But, for this graph G, it follows from Theorem 1.3 that the set $S = \{v_4, v_2, v_6, v_7, \ldots, v_p\}$ is a connected detour basis so that $cdn(G) = p - 3$. So, in this case there are no graphs satisfying the requirements of the theorem. If x is adjacent only to v_4, then we get a graph G isomorphic to the one given in Figure 2.8 and hence in this case also there are no graphs satisfying the requirements of the theorem. If x is adjacent to both v_2 and v_4, then the graph reduces to the one given in Figure 2.9.

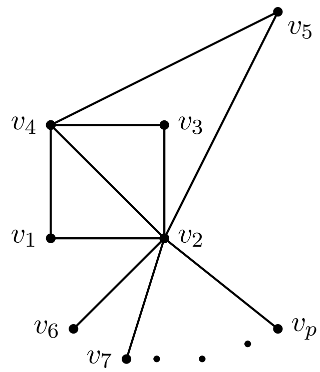


FIGURE 2.8. G

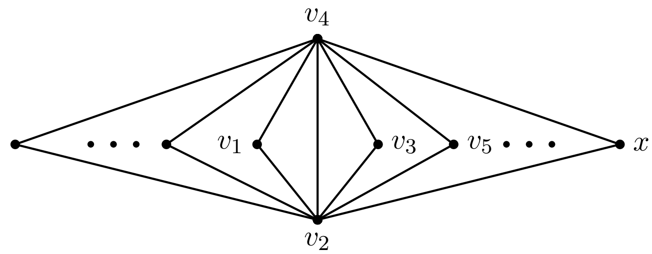


FIGURE 2.9. G

However for this graph, $\{v_2, v_4\}$ is a connected detour basis so that $cdn(G) = 2$ and hence $cdn(G) \leq p - 4$, which is a contradiction. Thus a vertex not in H_2 cannot be adjacent to both v_2 and v_4.

Next, if a vertex x not on H_2 is adjacent only to v_2 and a vertex y not on H_2 is adjacent only to v_4, then x and y must be pendant vertices of G, for otherwise, we get either a path or a cycle of length ≥ 5 so that $D \geq 5$, which is a contradiction. Thus in this case, the graph reduces to the one given in Figure 2.10. But, for this graph G, it follows from Theorem 1.3 that the set of all end-vertices together with the cut-vertices is a connected detour basis so that $cdn(G) = p - 3$. So, in this case also there are no graphs satisfying the requirements of the theorem. Thus we conclude that in this case also there are no graphs G with H_2 as proper subgraph. Hence we conclude that, if G does not contain K_4 as an induced subgraph, then G has a unique 4-cycle. Now we consider two subcases.

Subcase 1: The unique cycle $C_4 : v_1, v_2, v_3, v_4, v_1$ contains exactly one chord v_2v_4. Since $p \geq 5$, $D = 4$ and G is connected, any vertex x not on C_4 is pendant and is adjacent to at least one vertex of C_4. The vertex x cannot be adjacent to both v_1 and v_3, for in this case we get $c(G) = 5$, which is a contradiction. Suppose that x is adjacent to v_1 or v_3, say v_1. Also if y is a

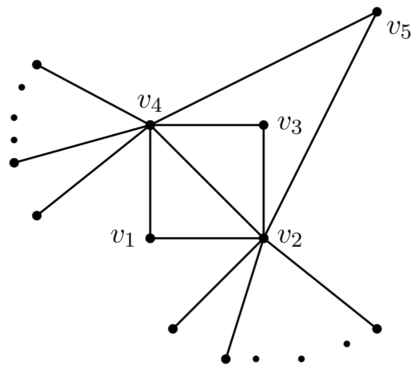


FIGURE 2.10. G

vertex such that $y \neq x, v_1, v_2, v_3, v_4$, then y cannot be adjacent to v_2 or v_3 or v_4, for in each case $D \geq 5$, which is a contradiction. Hence y is a pendant vertex and cannot be adjacent to x or v_2 or v_3 or v_4 so that in this case the graph G reduces to the one given in Figure 2.11.

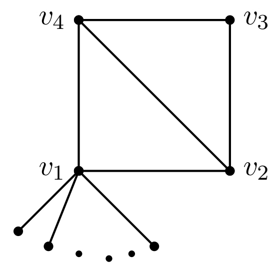


FIGURE 2.11. G

But, for this graph G, it follows from Theorem 1.3 that the set of all end-vertices together with the cut-vertex v_1 and the vertex v_2 forms a connected detour basis so that $cdn(G) = p - 2$. Similarly, if x is adjacent to v_3, then we get a contradiction.

Now, if x is adjacent to both v_2 and v_4, then we get the graph H_2 given in Figure 2.4 as a subgraph, where $x = v_5$. Then as in the first part of case 2, we see that there are no graphs which satisfy the requirements of the theorem.

Thus x is adjacent to exactly one of v_2 or v_4, say v_2. Also if y is a vertex such that $y \neq x, v_1, v_2, v_3, v_4$, then y cannot be adjacent to x or v_1 or v_3, for in each case $D \geq 5$, which is a contradiction. If y is adjacent to v_2 and v_4, then we get the graph H given in Figure 2.12 as a subgraph. Then exactly as in the first part of case 2 it can be seen that there are no graphs satisfying the requirements of the theorem.

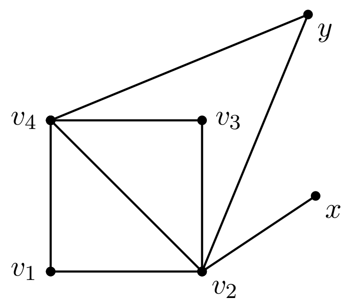
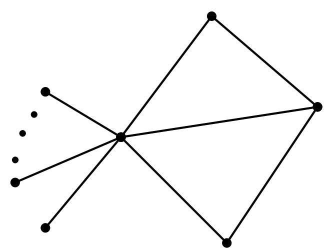
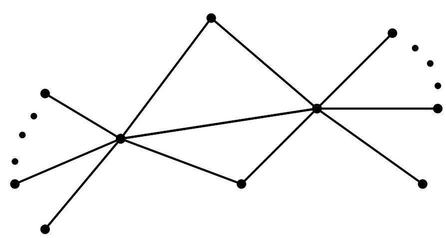


FIGURE 2.12. H

Thus y must be adjacent to v_2 or v_4 only. Hence we conclude that in either case the graph G must reduce to the graph G_1 or G_2 given in Figure 2.13. Similarly, if x is adjacent to v_4, then the graph G reduces to the graph G_1 or G_2 given in Figure 2.13.



G_1



G_2

FIGURE 2.13.

It follows from Theorem 1.3 that $cdn(G_1) = cdn(G_2) = p - 2$. Thus in this case there are no graphs satisfying the requirements of the theorem.

Subcase 2: The unique cycle $C_4 : v_1, v_2, v_3, v_4, v_1$ has no chord. In this case we claim that G contains no triangle. Suppose that G contains a triangle C_3. If C_3 has no vertex in common with C_4 or exactly one vertex in common with C_4, we get a path of length at least 5 so that $D \geq 5$. If C_3 has exactly two vertices in common with C_4, we get a cycle of length 5. Thus, in all cases, we have a contradiction and hence it follows that G contains a unique chordless cycle C_4 with no triangles. Since $p \geq 5, D = 4, c(G) = 4$ and G is connected, any vertex x not on C_4 is pendant and is adjacent to exactly one vertex of C_4, say v_1. Also if y is a vertex such that $y \neq x, v_1, v_2, v_3, v_4$, then y cannot be adjacent to v_2 or v_4, for in this case $D \geq 5$, which is a contradiction. Thus y must be adjacent to v_3 only. Hence we conclude that in either case G must reduce to the graphs H_1 or H_2 as given in Figure 2.14.

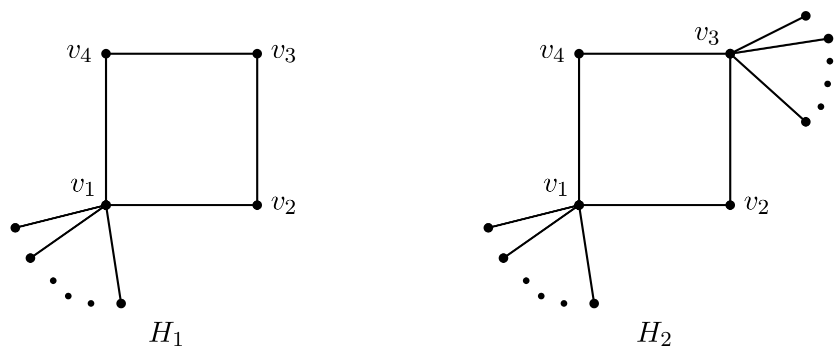


FIGURE 2.14.

But, for these graphs H_1 and H_2 in Figure 2.14, it follows from Theorem 1.3 that $cdn(H_1) = p - 2$ and $cdn(H_2) = p - 1$. Hence in this case also there are no graphs satisfying the requirements of the theorem. Thus, when $D = 4$ and $c(G) = 4$, there are no graphs satisfying the requirements of the theorem.

Case 3: Let $c(G) = 3$. First we prove that the graph contains at most two triangles. If G contains more than two triangles, since $D = 4$, it is clear that all the triangles must have a vertex v in common. Now, if two triangles have two vertices in common then it is clear that $c(G) \geq 4$. Hence all triangles must have exactly one vertex in common. Since $D = 4$, all the vertices of all the triangles are of degree 2 except v. Thus the graph reduces to the graphs given in Figure 2.15.

Now, it follows from Theorems 1.2 and 1.3 that the set S consisting of all the end-vertices, all the cut- vertices and exactly one vertex other than v from each of the triangles is a connected detour set of G so that $cdn(H_i) = |S|$ $(i = 1, 2)$. Since G contains more than two triangles it follows that $|S| \leq p - 3$ and so $cdn(H_i) \leq p - 3$ $(i = 1, 2)$, which is a contradiction to the assumption that

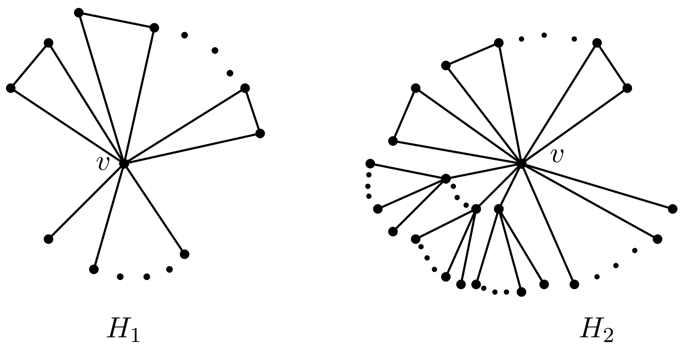


FIGURE 2.15.

$cdn(G) = p$. Thus the graph G contains at most two triangles. Now we consider two cases.

Case 3a: Suppose that G contains exactly one triangle $C_3 : v_1, v_2, v_3, v_1$. Since $p \geq 5$, there are vertices not on C_3. If all the vertices of C_3 have degree three or more, then since $D = 4$, the graph G must reduce to the graph G given in Figure 2.1. Now, by Theorem 1.4, $cdn(G) = p$ and so G is the only graph in this case satisfying the requirements of the theorem. Now, suppose that at most two vertices of C_3 have degree ≥ 3. We consider two subcases.

Subcase 1: Exactly two vertices of C_3 have degree 3 or more. Let deg_G $v_3 = 2$. Now, since $p \geq 5$, $D = 4$, $c(G) = 3$ and G is connected, we see that the graph reduces to the graph G_4 given in Figure 2.18 and it follows from Theorem 1.3 that $cdn(G_4) = p - 1$ and so in this subcase there are no graphs satisfying the requirements of the theorem.

Subcase 2: Exactly one vertex v_1 of C_3 has degree 3 or more. Since G is connected, $p \geq 5$, $D = 4$ and $c(G) = 3$, the graph reduces to the one given in Figure 2.16.

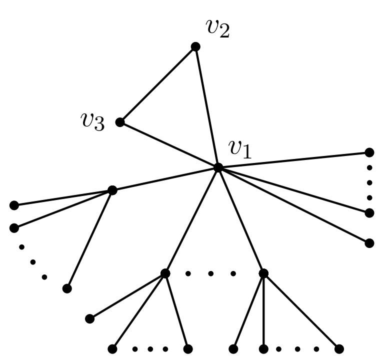


FIGURE 2.16. G

Now, it follows from Theorems 1.2 and 1.3 that the set of all the end-vertices together with the cut-vertices and the vertex v_2 is a connected detour basis so that $cdn(G) = p - 1$. So in this subcase also there are no graphs satisfying the requirements of the theorem. Thus we conclude that the graph G given in Figure 2.1 is the only graph in this case satisfying the requirements of the theorem.

Case 3b: Suppose that G contains exactly two triangles. Since G is connected, $p \geq 5$, $c(G) = 3$ and $D = 4$, the two triangles do not have two vertices in common and the graph G reduces to G_1, G_2 or G_3 as given in Figure 2.17.

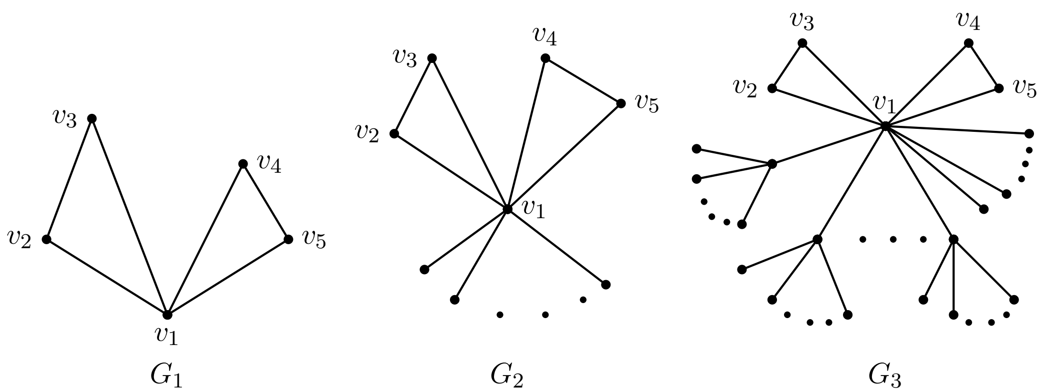


FIGURE 2.17. G

Now, if G is one of these G_i $(i = 1, 2, 3)$, then it follows from Theorems 1.2 and 1.3 that the set S of all the all end-vertices together with the cut-vertices and the vertices v_2 and v_4 is a connected detour basis of G so that $cdn(G) = p - 2$. Thus in this case also there are no graphs satisfying the requirements of the theorem. Hence, we conclude that the graph G given in Figure 2.1 is the only graph in this case satisfying the requirements of the theorem. This completes proof of the theorem. ■

In view of Theorem 2.1, we leave the following problem as an open question.

Problem 2.2. *Characterize connected graphs G with detour diameter $D \geq 5$ for which $cdn(G) = p$.*

Next we characterize graphs G with detour diameter $D \leq 4$ for which $cdn(G) = p - 1$.

For this purpose we introduce the collection $\mathscr{L}$ of graphs given in Figure 2.18.

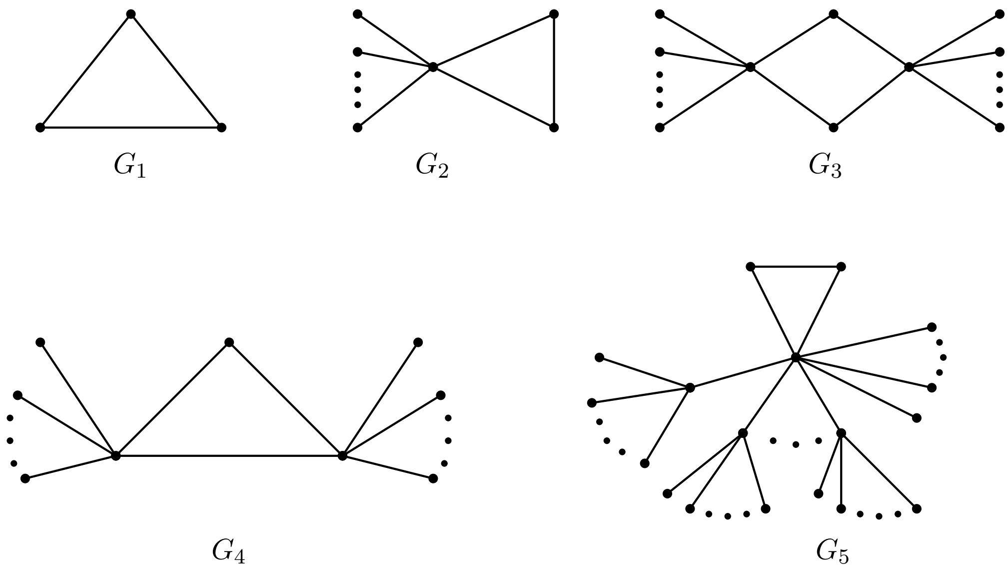


FIGURE 2.18. Graphs in family $\mathscr{L}$

Theorem 2.3. *Let G be a connected graph of order $p \geq 3$ with detour diameter $D \leq 4$. Then $cdn(G) = p - 1$ if and only if $G \in \mathscr{L}$ as given in* Figure 2.18.

Proof. It is straightforword to verify that $cdn(G_i) = p - 1$ $(1 \leq i \leq 5)$ for all the graphs $G_i \in \mathscr{L}$ given in Figure 2.18, using Theorems 1.2, 1.3, 1.6 and 1.9.

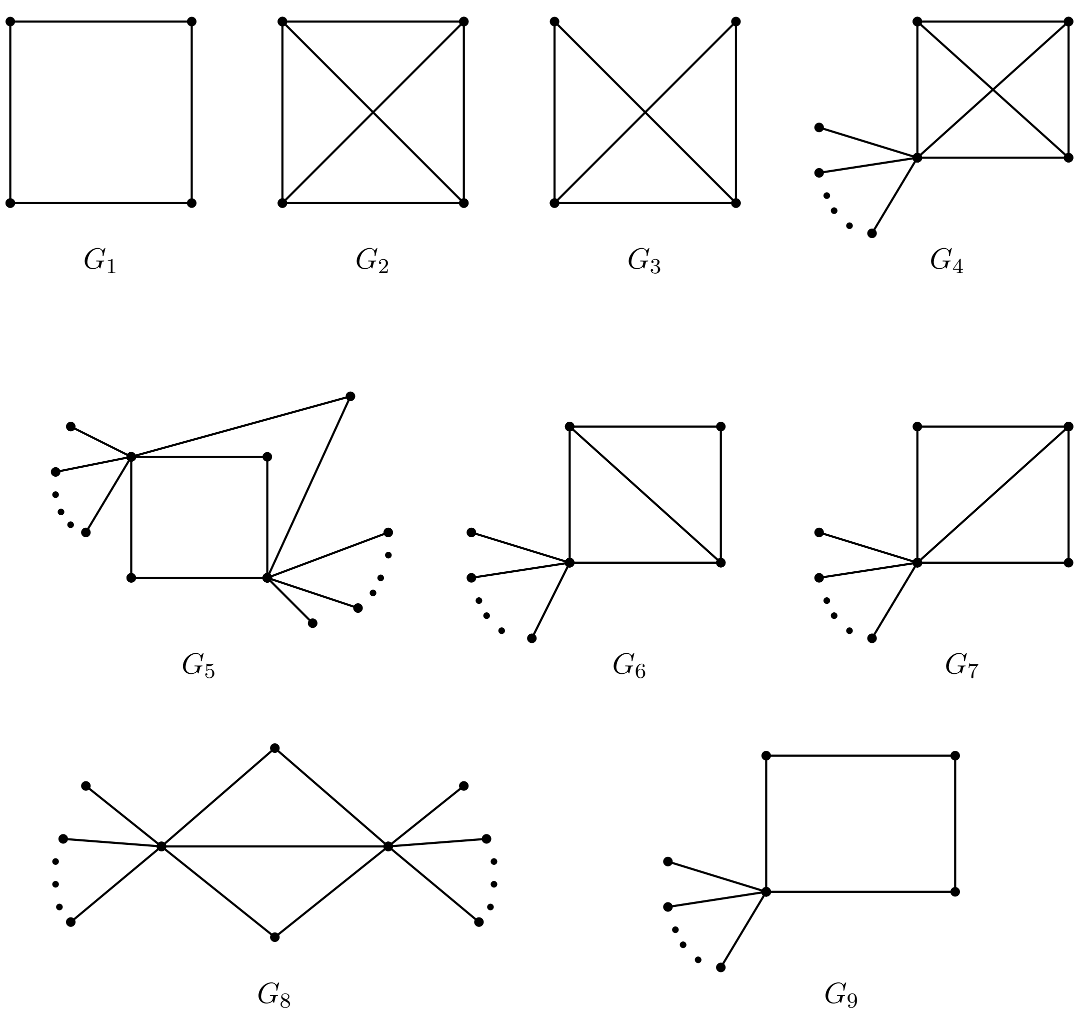
For the converse, let G be a graph of order $p \geq 3$, $D \leq 4$ and $cdn(G) = p-1$. Then by an argument as in Theorem 2.1 it is seen that the graph reduces to the graphs $G_i \in \mathscr{L}\,(1 \leq i \leq 5)$ as given in Figure 2.18 This completes the proof of the theorem. ∎

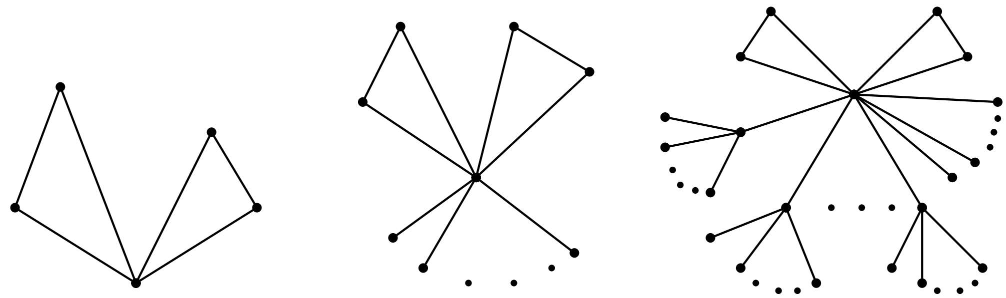
In view of Theorem 2.3, we leave the following problem as an open question.

Problem 2.4. *Characterize connected graphs G with detour diameter $D \geq 5$ for which $cdn(G) = p - 1$.*

Next we characterize graphs G with detour diameter $D \leq 4$ for which $cdn(G) = p - 2$.

For this purpose we introduce the collection $\mathscr{F}$ of graphs given in Figure 2.19.


G_1
G_2
G_3
G_4
G_5
G_6
G_7
G_8
G_9



G_{10} G_{11} G_{12}

FIGURE 2.19. Graphs in family $\mathscr{F}$

Theorem 2.5. *Let G be a connected graph of order $p \geq 4$ with detour diameter $D \leq 4$. Then $cdn(G) = p - 2$ if and only if $G \in \mathscr{F}$ given in* Figure 2.19.

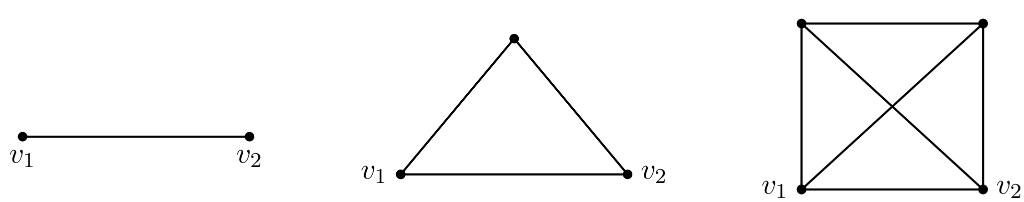
Proof. It is straightforword to verify that $cdn(G_i) = p - 2$ $(1 \leq i \leq 12)$ for all the graphs $G_i \in \mathscr{F}$ given in Figure 2.19, using Theorems 1.2, 1.3, 1.6 and 1.9.

For the converse, let G be a graph of order $p \geq 4$, $D \leq 4$ and $cdn(G) = p-2$. Then, by an argument as in Theorem 2.1, it is seen that the graph reduces to the graphs $G_i \in \mathscr{F}$ $(1 \leq i \leq 12)$ as given in Figure 2.19. This completes the proof of the theorem. ∎

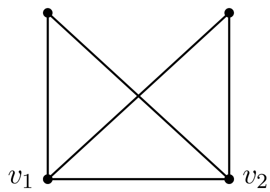
In view of Theorem 2.5, we leave the following problem as an open question.

Problem 2.6. *Characterize connected graphs G with detour diameter $D \geq 5$ for which $cdn(G) = p - 2$.*

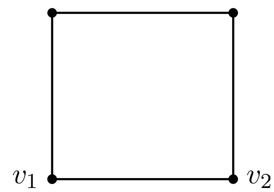
In the following we characterize graphs G with detour diameter $D \leq 4$ for which $cdn(G) = 2$. For this purpose we introduce the collection $\mathscr{R}$ of graphs given in Figure 2.20.



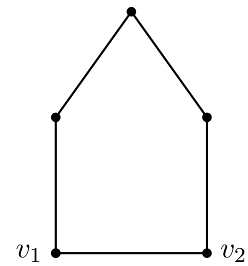
G_1 G_2 G_3



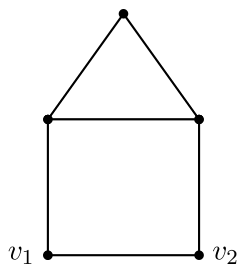
G_4



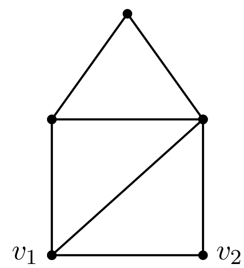
G_5



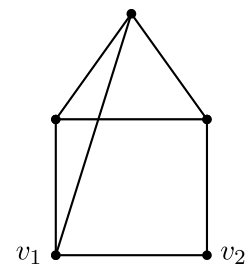
G_6



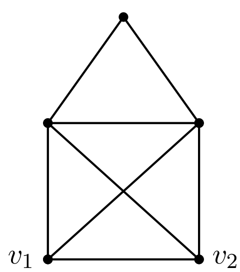
G_7



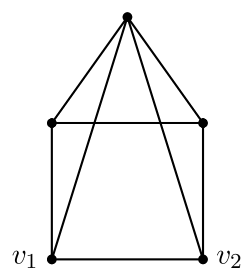
G_8



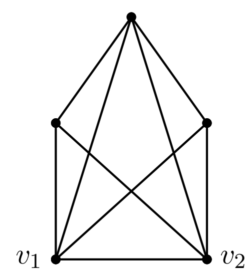
G_9



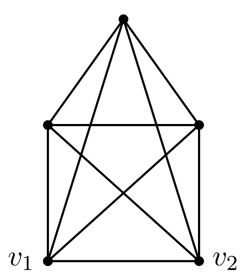
G_{10}



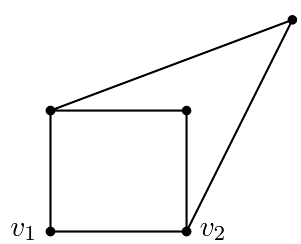
G_{11}



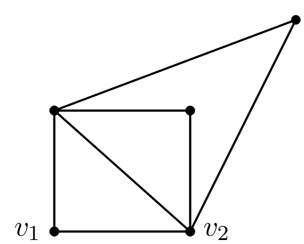
G_{12}



G_{13}



G_{14}



G_{15}

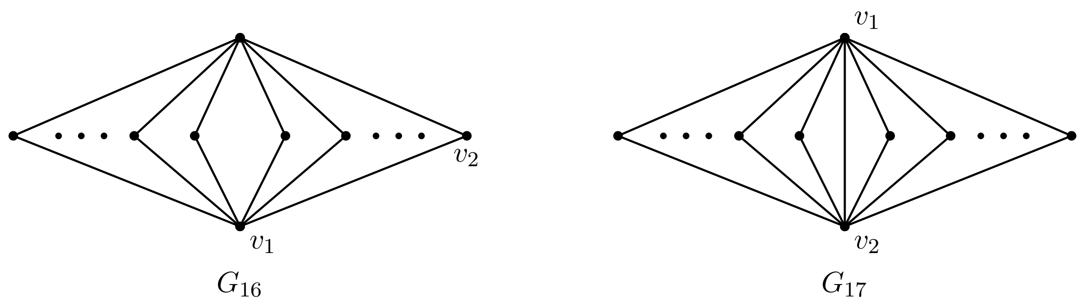


FIGURE 2.20. Graphs in family $\mathscr{R}$

Theorem 2.7. *Let G be a connected graph of order $p \geq 2$ with detour diameter $D \leq 4$. Then $cdn(G) = 2$ if and only if $G \in \mathscr{R}$ given in* Figure 2.20.

Proof. It is straightforword to verify that $cdn(G_i) = 2 \quad (1 \leq i \leq 17)$ for all the graphs $G_i \in \mathscr{R}$ given in Figure 2.20, using Theorems 1.5, 1.7 and 1.9

For the converse, let G be a graph of order $p \geq 2$, $D \leq 4$ and $cdn(G) = 2$. Then by an argument as in Theorem 2.1 it is seen that the graph reduces to the graphs $G_i \in \mathscr{R}$ $(1 \leq i \leq 17)$ as given in Figure 2.20. This completes the proof of the theorem. ∎

In view of Theorem 2.7, we leave the following problem as an open question.

Problem 2.8. *Characterize connected graphs G with detour diameter $D \geq 5$ for which $cdn(G) = 2$.*

A connected detour set S in a connected graph G is called a *minimal connected detour set* of G if no proper subset of S is a connected detour set of G. The *upper connected detour number* $cdn^+(G)$ of G is the maximum cardinality of a minimal connected detour set of G. It is clear that $cdn(G) \leq cdn^+(G)$ for any connected graph G and it is proved in [5] that for every pair a, b of integers with $5 \leq a \leq b$, there exists a connected graph G with $cdn(G) = a$ and $cdn^+(G) = b$.

In the rest of the paper, we introduce the forcing connected detour number of a graph G, determine its properties and prove a realization result with regard to the connected detour number and the forcing connected detour number of a graph G.

3. Forcing Subsets in Connected Detour Sets

Definition 3.1. *Let G be a connected graph and S a connected detour basis of G. A subset $T \subseteq S$ is called a* forcing subset *for S if S is the unique connected detour basis containing T. A forcing subset for S of minimum cardinality is a* minimum forcing subset *of S. The* forcing connected detour number *of*

S, denoted by $fcdn(S)$, *is the cardinality of a minimum forcing subset for S. The* forcing connected detour number *of G, denoted by* $fcdn(G)$, *is* $fcdn(G) = min\{fcdn(S)\}$, *where the minimum is taken over all connected detour bases S in G.*

The following theorem is an easy consequence of the definitions of the connected detour number and the forcing connected detour number of a connected graph G.

Theorem 3.2. *For every connected graph G,* $0 \leq fcdn(G) \leq cdn(G)$.

Example 3.3. *For the graph G given in Figure 3.1,* $S_1 = \{u, s, w, t, v\}$ *is the unique connected detour basis of G so that* $fcdn(G) = 0$ *and for the graph G given in Figure 1.1,* $S_2 = \{v_1, v_2, v_3\}$ *and* $S_3 = \{v_1, v_2, v_5\}$ *are the only connected detour bases of G so that* $cdn(G) = 3$ *and* $fcdn(G) = 1$.

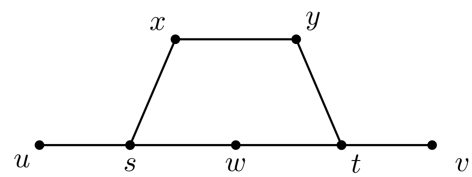


FIGURE 3.1. G

The following theorem gives the forcing number of certain graphs G.

Theorem 3.4. a) *If G is the complete graph* K_p $(p \geq 3)$ *or the the cycle* C_p *or the complete bipartite graph* $K_{m,n}$ $(m, n \geq 2)$, *then* $cdn(G) = fcdn(G) = 2$.
b) *If G is a tree of order* $p \geq 2$, *then* $cdn(G) = p$ *and* $fcdn(G) = 0$.

Proof. a) By Theorem 1.8, a set S of vertices is a connected detour basis of G if and only if S consists of two adjacent vertices of G. For each vertex v in G there are at least two vertices adjacent with v. Thus the vertex v belongs to more than one connected detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any connected detour basis of G. Thus the result follows.
b) By Theorem 1.5, $cdn(G) = p$. Thus the set of all vertices of a tree is the unique connected detour basis so that $fcdn(G) = 0$. ∎

In view of Theorem 3.2, the following theorem gives a realization result.

Theorem 3.5. *For each pair a, b of integers with* $0 \leq a < b$ *and* $b \geq 3$, *there is a connected graph G with* $fcdn(G) = a$ *and* $cdn(G) = b$.

Proof. **Case 1:** $a = 0$. For each $b \geq 3$, let G be a tree with b vertices. Then $fcdn(G) = 0$ and $cdn(G) = b$ by Theorem 3.4(b).
Case 2: $a \geq 1$. For each integer i with $1 \leq i \leq a$, let F_i be a copy of the complete graph K_2, where $V(F_i) = \{u_i, v_i\}$ and let $H = K_{1,b-a-1}$ be the

star whose vertex set is $W = \{z_1, z_2, \ldots, z_{b-a-1}, v\}$. Then the graph G is obtained by joining the central vertex v of H to the vertices of $F_1, F_2, \ldots, F_a$. The graph G is connected and is shown in Figure 3.2. Then by Theorem 1.6, $cdn(G) = b$.

Now, we show that $fdn(G) = a$. It is clear that W is the set all connected detour vertices of G. Let U be any connected detour basis of G. Then $cdn(G) = |U|$, $W \subseteq U$ and U is the unique connected detour basis containing $U - W$. Hence $fcdn(G) \leq |U - W| = |U| - |W| = cdn(G) - |W| = b - (b - a) = a$. Now, since $cdn(G) = b$, it follows from Theorem 1.2 that any connected detour basis of G is of the form $S = W \cup \{x_1, x_2, \ldots, x_a\}$, where $x_i \in \{u_i, v_i\}$ $(1 \leq i \leq a)$. Let T be a subset of S with $|T| < a$. Then there is a vertex x_j $(1 \leq j \leq a)$ such that $x_j \notin T$. Let y_j be a vertex of F_j distinct from x_j. Then $S' = (S - \{x_j\}) \cup \{y_j\}$ is also a connected detour basis such that it contains T. Thus S is not the unique connected detour basis containing T and so T is not a forcing set of S. Since this is true for all connected detour bases of G, it follows that $fcdn(G) \geq a$ and so $fcdn(G) = a$. ■

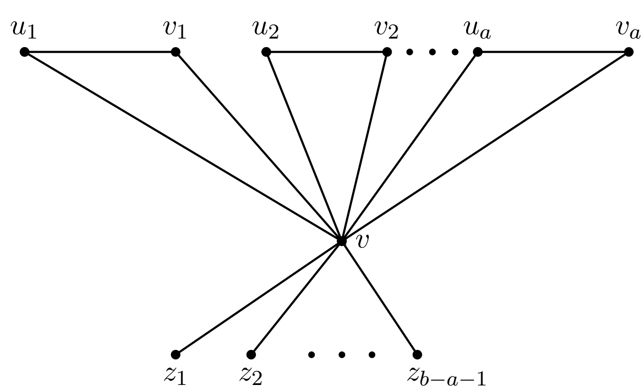


FIGURE 3.2. G

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