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METHOD OF THE EFFECTIVE POTENTIAL FOR THE FACTORIZED RAPIDLY CHANGING FORCE

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ABSTRACT. We propose the general form of Kapitza's effective potential method for a classical particle in the external filed of a rapidly changing open-loop control force with factorized dependency on the spatial and time coordinates. The result is illustrated by the application to stabilize the harmonic oscillator in non-trivial stable point.

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The set of open-loop control methods is very wide and difficult to systematize to compare with feedback control [1]. An important part of it is a sub-set of the methods based on the rapidly changing external forces [1,2]. Here we generalize the famous Kapitzas effective potential method [3,4] to the form allowing an arbitrary dependency of the open-loop control signal on the time, that includes the particular cases of harmonic modulation [3] and Fourier-expanded periodical modulation [5].

We apply the model of a classical particle with the mass m in two fields: one is a time-independent potential U, and another is rapidly changing open-loop control field $f(x,t)$ with a characteristic time of changing T. For the simplicity we will discuss one-dimensional movement.

As it has been shown in [3], the dynamical equations for the particle can be presented as a superposition $x = X + \xi$ of the smooth motion X and the fast

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changing component ξ:

$$m\ddot{X} = -\frac{dU(X)}{dX} - \xi\frac{d^2U(X)}{dX^2} + \xi\frac{df(X,t)}{dX} \ ;$$
$$m\ddot{\xi} = f(X,t) \ . \tag{1}$$

After the application of Kapitza's averaging procedure [4]:

$$< ... >= \frac{1}{T}\int_0^T ... \ dt$$

the second term in RHS (1a) becomes zero as $< \xi >= 0$. Then Eq. (1a) can be presented in the form:

$$m\ddot{X} = -\frac{dU(X)}{dX} + \left\langle \xi\frac{df(X,t)}{dX} \right\rangle \ . \tag{2}$$

Now let's suppose $\xi(0) = 0$ and $\dot{\xi}(0) = 0$. In [5] we studied the periodical force expanded in time Fourier series. Here we propose the factorized form for the force:

$$f(x,t) = \sum_k F_k(x)G_k(t) \ , \tag{3}$$

where F_k are the spacial components of the force, and G_k are dimensionless functions of the time. From (1b) and (3) we get:

$$\xi(t) = \frac{1}{m}\sum_k F_k(X)\int dt \ \tilde{G}_k(t) \tag{4}$$

with the notation:

$$\tilde{G}_k(t) = \int dt \ G_k(t) \ . \tag{5}$$

We substitute (4) to calculate the time average in RHS (2). It becomes:

$$\xi\frac{df}{dX} = \frac{1}{m}\sum_{k,l} F_k(X)\frac{dF_l(X)}{dX}\int dt \ \tilde{G}_k(t)G_l(t)$$

and

$$\left\langle \xi\frac{df}{dX} \right\rangle = \frac{1}{m}\sum_{k,l} F_k(X)\frac{dF_l(X)}{dX}\cdot\frac{1}{T}\int_0^T dt\int dt \ \tilde{G}_k(t)G_l(t) \ .$$

But $G_l(t) = d\tilde{G}_l(t)/dt$, so

$$\int_0^T\int dt \ \tilde{G}_k(t)\frac{d\tilde{G}_l(t)}{dt} = \left[\tilde{G}_l(t)\int dt \ \tilde{G}_k(t)\right]_0^T - \int_0^T dt \ \tilde{G}_k(t)\tilde{G}_l(t) \ .$$

By (4) the first term of RHS here can be combined into $\xi(X,t)$. Let's suppose $\xi(X,0)=\xi(X,T)=0$ to exclude it. Then

$$\left\langle \xi \frac{df}{dX} \right\rangle = -\frac{1}{m}\sum_{k,l} F_k(X)\frac{dF_l(X)}{dX}\cdot\frac{1}{T}\int_0^T dt\ \tilde{G}_k(t)\tilde{G}_l(t) =$$

$$= -\frac{1}{m}\sum_{k,l} F_k(X)\frac{dF_l(X)}{dX}\left\langle \tilde{G}_k(t)\tilde{G}_l(t)\right\rangle = -\frac{1}{m}\sum_{k,l} F_k(X)\frac{dF_l(X)}{dX}G_{kl} \qquad (6)$$

with the notation

$$G_{kl} = \left\langle \tilde{G}_k(t)\tilde{G}_l(t)\right\rangle \qquad (7)$$

for the correlator. RHS (6) is not symmetric with respect to its indices k,l, but the correlator (7) is. We will use this property $G_{kl}=G_{lk}$ to re-write the average (6) in the symmetric form:

$$\left\langle \xi \frac{df}{dX} \right\rangle = -\frac{1}{2m}\sum_{k,l}\left(F_k(X)\frac{dF_l(X)}{dX}+\frac{dF_k(X)}{dX}F_l(X)\right)G_{kl}\ ,$$

or

$$\left\langle \xi \frac{df}{dX} \right\rangle = -\frac{1}{2m}\sum_{k,l}\frac{d(F_k(X)F_l(X))}{dX}G_{kl}\ .$$

Finally we present (2) as:

$$m\ddot{X} = -\frac{dU(X)}{dX} - \frac{d}{dX}\left(\frac{1}{2m}\sum_{k,l}G_{kl}F_k(X)F_l(X)\right) \equiv -\frac{dU_{eff}(X)}{dX}\ . \qquad (8)$$

Thus, the effective potential energy of the system can be define as:

$$U_{eff} = U + \frac{1}{2m}\sum_{k,l}G_{kl}F_k(X)F_l(X)\ . \qquad (9)$$

Eqs. (5), (7) and (9) include as particular cases the sin- and cos-shapes for the time dependency from [3] and the Fourier expansion presented in [5]. If G_k depends only on the dimensionless time value $\tau = t/T$, then $G_k = G_k(\tau)$,

$$\tilde{G}_k(t) = \int dt\ G_k(t) = T\int d\tau\ G_k(\tau) \equiv T\cdot\tilde{G}_k(\tau)$$

and

$$G_{kl} = T^2\int_0^1 d\tau\ \tilde{G}_k(\tau)\tilde{G}_l(\tau)\ , \qquad (10)$$

i.e. the correlator G_{kl} has the second order with respect to the small time T (characteristic time of the open-loop control changing). To be of the same order as U the second term in RHS (9) must have a factor of the order T^{-2} in the F-force product. It means that every force component F_k should be of the order $1/T$.

Now let's illustrate the application of the general result (9) to stabilize harmonic classical oscillator with the potential $U = \alpha X^2/2$, where α is a positive constant, in a non-trivial stable point. We suppose the linear force with $k, l = 1, 2$:

$$F_1 = \frac{\beta_1 \sqrt{m}}{T} \cdot L \ ; \quad F_2 = \frac{\beta_2 \sqrt{m}}{T} \cdot T \ ;$$

β_1, β_2, L=const> 0, (we apply L just to have the same dimension for β_1, β_2) and define:

$$g_{kl} = \int_0^1 d\tau \ \tilde{G}_k(\tau) \tilde{G}_l(\tau) \ . \tag{11}$$

Then we write down the effective potential:

$$U_{eff} = \frac{\alpha X^2}{2} + \frac{1}{2} \left(\beta_1^2 g_{11} L^2 + 2\beta_1 \beta_2 g_{12} L X + \beta_2^2 g_{22} X^2 \right) \ . \tag{12}$$

The stable points of the oscillator are defined by minima X_* of its effective potential (12):

$$\frac{dU_{eff}(X_*)}{dX} = (\alpha + \beta_2^2 g_{22}) X_* + \beta_1 \beta_2 g_{12} L = 0$$

and

$$\frac{d^2 U_{eff}(X_*)}{dX^2} = \alpha + \beta_2^2 g_{22} > 0$$

The last condition is always true as $g_{22} > 0$. Then the stable point of the oscillator corresponds to the minimum:

$$X_* = -\frac{\beta_1 \beta_2 g_{12} L}{\alpha + \beta_2^2 g_{22}} \ . \tag{13}$$

In fact the sign of the minimum (13) is defined from the sign of g_{12}, that in its turn is defined by (11) with the shapes of open-loop control signal (5). Thus, the shape of the control signal defines the position of the equilibrium point.

The same method can be applied to investigate the stabilities of more complicate physical systems, like Kapitza pendulum with rapidly oscillating pivot [5], cooled atom beam in the field of modulated optical wave(s) [6,7] etc.

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