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EULER'S TRACE IN MATHEMATICS. TO EULER'S 300TH BIRTHDAY

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This year mathematicians celebrate the 300th anniversary of Leonhard Euler (1707–83). Let us recall some history[1]

Scientists of St. Petersburg were contemplating celebration of Euler's 200th birthday as early as in the last year of XIX century, long ahead of its due time. On February 6, 1899 the General Assembly of the Russian Academy of Sciences discussed the proposal of the Department of Physics and Mathematics to organize an international subscription for a monument to Euler to be erected in St. Petersburg. Academician (in mathematics) N. Ya. Sonin (1849–1915) spoke strongly against it.

He declared that Euler's works were out-of-date, that Euler was surpassed by Lagrange and Gauss, and that "the traces of Euler's activity in mathematics are effectively covered up". Monuments, he said, were to be raised to commemorate the greatest scientists while Euler had been at most a very good one, therefore a bust for him in the conference hall of the Academy was quite sufficient (a bust had already been set up shortly after Euler's death). Some of the Academicians showed that they failed to understand why the monument was to be located necessarily in St. Petersburg and not, say, in Basel, Euler's

[1] *Translator's note*: The following essay was written in 1984. Its expanded and modified version is a part of a wonderful book *Tales on physicists and mathematicians* published in Russian in huge circulation (http://www.mccme.ru) whose first edition was translated into English, French, Japanese, and more.
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birth-place, or in Berlin, where he had been working for almost as long time as in St. Petersburg.

The proposal was put to the vote and the votes were equally divided. According to democratic rules of the Academy this meant the rejection of the offer.

In the years to follow, however, Euler's reputation as a greatest scientist seems to have been firmly established (although no monument had ever been erected to him!). This anecdote could, therefore, be easily regarded as a sheer curiosity (which, it was, without doubt); one could say that N. Ya. Sonin was hardly a competent judge of Euler's greatness, but all the same...

Let us try to understand which features of Euler's activity might have provided a ground for such a staggering conclusion. What constitutes the greatness of Euler? What trace in mathematics did he leave and has it really been "covered up"?

Euler is perhaps a forerunner in the number of objects named after him in mathematics. One has *Euler's function*, *Euler's constant*, *Euler's line* (in the geometry of a triangle), *Euler's integrals* (Gamma- and Beta-functions), and many more. There is also *Euler's formula* for a polyhedron, no less remarkable *Euler's formula* relating the exponential and the trigonometric functions, the *Euler-Lagrange equation* in the Calculus of Variations, a lot of other theorems and constructions bearing his name — it seems impossible to list them all...

We might also recall that it was Euler who introduced e (Napier's number), $i = \sqrt{-1}$, the symbol $f(x)$, and the modern notation for the trigonometric functions (and it was again Euler who first began to consider them outside the segment $[0, 2\pi]$).

Of course, one could always say that the last series of his innovations was a purely methodological achievement. Acad. A. A. Markov (1856–1922), the principal opponent of N. Ya. Sonin at the Academy gathering in 1889, reminded the audience that Euler's ideas had still been employed in teaching. N. Y. Sonin retorted that this only showed underdevelopment of the educational system.

Nowadays, Euler's name is a very often cited one. But what are the great theories for which he is credited in the history of mathematics, what difficult (and for a long time unsolved) problems did he manage to resolve (this being so esteemed in our century)?

Looking into all that in more detail one realizes that Euler's case is not a simple one. A proper name is often assigned to a discovery in a random and an unjust way. Often it is only the name of a person who had made the last step or even of a one who had explained the result more clearly to others, that survives. In Sonin's terminology, the traces of those who have made first, not necessarily decisive, but important and difficult steps are usually "covered up" by their successors.

It was not customarily for Euler to finish works started by someone else. Still more significant is that some of his most popular results did "cover up the traces" of his predecessors.

A preliminary version of *Euler's theorem* — the relation

$$V + F = E + 2$$

between the numbers of edges, E, faces, F, and vertices, V, of a given convex polyhedron — is contained in the manuscript by René Descartes (1596–1650) dated by 1639. The theorem was copied by G. W. Leibniz (1646–1716). Descartes' statement sank into oblivion; Euler formulates the theorem now named after him in a letter to Ch. Goldbach (1690–1764) in 1750. Euler had soon found a proof of the theorem he first tested on examples.

In 1730, Euler constructed two remarkable functions, $\Gamma(x)$ and $B(x, y)$, which at integer points coincide, respectively, with $n!$ and $\dfrac{(n+m)!}{n!m!}$, and gave their representation by means of integrals. Doing so, he was well aware of the ideas of J. Wallis (1616–1703) from Wallis's "Arithmetic of Infinites" (1656), and has used integrals depending on continuous parameters to approximate numerical sequences. J. Wallis had, to a large extent, mastered the Euler integral of the first kind, $B(x, y)$, and knew of its connection with the decomposition of π into an infinite product.

J. L. Lagrange (1736–1813) had called Euler's formula

$$e^{ix} = \cos x + i \sin x$$

"one of the most beautiful analytic discoveries of our century". For anyone who first encounters it, the formula does produce a striking impression even today. A natural way to obtain it involves either power series or functional equations, and hence one hardly recalls how it had been discovered in mathematics of XVIII century. It is miraculous that the logic of the discovery was quite straightforward. At the beginning of the century, Johann I Bernoulli (1667–1748), Euler's teacher, investigating the problem of integrating rational functions, noted the relation

$$\frac{1}{1+x^2} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

After a formal integration it yields the arctangents on the left-hand side and two logarithms (of an imaginary argument) on the right-hand side. A simple transformation then gives

$$x = \frac{1}{2i} \ln \frac{1 - i\tan x}{1 + i\tan x} \tag{1}$$

and this can be trivially transformed into the Euler formula.
Actually, Bernoulli's work does include (1), although he unsuccessfully tried to make sense of other computations with imaginary quantities. His attempts led to the well-known discussion Bernoulli had in 1712–13 with Leibniz which centered on the meaning of the logarithms of negative numbers (including, for example, the question: What is the value of $\ln(-1)$?).

In 1714, "Euler's formula" flickered, without necessary justifications, in a work of Roger Coats (1682–1716), Newton's colleague who died at an early age. Euler had been well-informed of the problems that troubled his teacher; in 1728, using his computations, he deduced formula (1), and by 1739 he developed a theory of logarithms in the complex domain which made all formulas well-defined and was free of contradictions (e.g., the value of $\ln(-1)$ was defined to be any element of the infinite set $(2k+1)\pi i$, where k is any integer).

That was one of Euler's greatest achievements.

It is a good example of Euler's ability to continue and finish on the highest level the works his predecessors were unable to conclude. In his publications be never failed to provide references to their works, but nevertheless, today we almost unreservedly connect these results with Euler's name only, and rarely mention those who had made the first steps.

Let us now turn to Euler's proficiency in solving concrete mathematical problems. One should realize, however, that there were times when difficult problems did not enjoy such a high prestige among mathematicians as they do now. True, in the mathematics of antiquity the geometrical construction problems for a ruler and compass (the squaring of a circle, the trisection of an angle, the doubling of a cube, the construction of a regular heptagon) were always considered to be of outstanding importance. These problems have been finally dealt with only in XIX century.

Another difficult problem — that of finding a solution to an arbitrary 3rd order equation — played an important role in reviving the mathematical science after a period of relative stagnation it experienced during the Middle Ages. This happened in XVI century; on the other hand, although the mathematicians of XVII and XVIII centuries never forgot that no formula for solving equations of degrees greater than 4 had been found, the problem did not define then the mainstream of the development of mathematics.

In XVII century, the century of Analysis, mathematical duels went out of fashion but a spirit of competition remained still: Mathematicians willingly exchanged their problems. One can recall that a series of problems dealing with the cycloid played an important role in the formation of the analysis of infinitesimals (Calculus).

The greatest mathematicians, however, were more occupied with developing new theories and solving problems which they had formulated themselves. But these were not problems which for decades challenged ambitions of scientists, not the problems one had to concentrate on for several years, devise intricate arguments which would have consisted of many steps and would have filled tens, even hundreds pages.

Euler was, probably, the first great mathematician who had a potential that made it possible for him to devote himself to solving problems of this kind. How did he realize it?

At that time the main source of difficult problems was the number theory. Setting its early history connected with the name of Diophantos (III century AD) apart, one can say that Pierre Fermat (1601–1665) was the first to discover that arithmetic contain not only wonderful facts about specific numbers but also general statements — theorems. He left formulations of a considerable number of such theorems on the margins of Diophantos' "Arithmetics" (issued pat to the purpose in 1621), as well as in his letters and notes.

Fermat was one of the greatest mathematicians of his time. He took the central part in the heroic epopee of creation of Calculus and Analytic Geometry and he maintained regular correspondence with many leading mathematicians. However, he was unable to excite an active interest in arithmetic problems among the most famous of his colleagues. Only two of his contemporaries of less caliber got interested: Frenicles de Bessi (1605–1675) and Jacques de Belli (1602–1679).

Mysteriously, some scientific theories, e.g., analysis in XVII century, attract almost every one while other theories are cultivated by rare individuals who try, in vain, to draw attention of their colleagues. The projective geometry, for example, was created by J. Desargues (1591–1661) and B. Pascal (1623–1662), both highly reputed scientists. Then it had been forgotten for a century and a half and rediscovered by G. Monge (1746–1818) and his pupils.

In 1770's Fermat's notes were partly collected and published, but the fate of his arithmetic would have been very doubtful if Euler never saw them.

P. L. Chebyshev (1821–1849) wrote in 1849:

Euler had started all investigations which now constitute the general number theory. His predecessor in these investigations was Fermat... But the discoveries of that geometrician had no direct influence on the development of the science: His propositions had neither proofs nor applications.

In such a condition Fermat's discoveries could only serve as a challenge to mathematicians to begin investigations in number theory. But whatever interest those investigations might have excited, no one dared to answer the call until Euler. And that can hardly surprise anyone: The investigations were not of the kind that would have required only the application of methods already known or development of tricks used earlier; it was impossible without inventing new tricks, discovering new sources, in one word — one had to establish a new science.

And all this had been done by Euler.

It seems that Euler first heard about Fermat's works soon after his arrival to St. Petersburg in 1727 from Ch. Goldbach (1690–1764) and then he kept interest in the number theory for the whole of his life. For a mathematician raised in the school of Johann I Bernoulli, to develop a stable interest in arithmetic was no small matter.

The most outstanding of Euler's colleagues considered his new folly occupation without understanding, to say the least. D. Bernoulli (1700–1782), who himself was not immune from occasional interest in arithmetical problems, wrote in 1778 in a letter to N. E. Fuss (1755–1826), an Euler's pupil, *a propos* arithmetic works of Fuss's teacher: "Don't you find that the primes are being treated with an excessive honor, too much effort being squandered for their sake and does it not reflect a refined taste of our century?"

Euler discussed arithmetical problems, first of all with Ch. Goldbach, a very original mathematician, who nevertheless, did not belong to the list of Euler's greatest contemporaries (such as Jean Le Rond d'Alembert (1717–1783) or A. C. Clairaut (1713–1765)). The situation improved only towards the end of Euler's life when, due to his works, the attitude to the number theory began to change and he had an opportunity to discuss these problems with Lagrange (their letters of 1772–73).

It was already in 1729, that Euler learned (from Goldbach) Fermat's claim that the numbers $F_n = 2^{2^n} + 1$ are prime for all n. In the beginning of 1730s, Euler found that the statement was false, namely, that F_5 is divisible by 641. Euler's discovery was not a result of a case-by-case checking: It would have been unrealistic to directly seek divisors of F_5 even for such an outstanding calculator as Euler. He had first found that all divisors of F_n (if any exist) had a very special form, namely, $k \cdot 2^{n+2} + 1$, and then there was no difficulty to pinpoint $641 = 5 \cdot 2^{5+2} + 1$. It is striking that Euler's first attempt to prove Fermat's claim had resulted in locating the only one which was false. Fortunately, it did neither discredit nor diminish Euler's interest in Fermat's arithmetics.

In 1736, after several unsuccessful attempts, Euler found a proof of the theorem which is now called the *small Fermat's theorem*: *If p is a prime and a is not divisible by p, then $a^{p-1} - 1$ is divisible by p* (e.g., let $a = 10$ and $p \neq 2, 5$, then the integer expressed by $p - 1$ nines is divisible by p).

Later Euler learned that Leibniz was already able to prove this statement (but Leibniz did not pursue his study of arithmetics any further: He saw no ways to construct its general theory, while individual problems never interested him very much).

Later in his life, Euler returned several times to the small Fermat's theorem. In 1773, he reformulated it replacing a prime p by any positive integer m. It turned out that $p - 1$ should then be replaced by $\varphi(m)$ (the *Euler function* which at point m is equal to the number of positive integers smaller than m and relatively prime to m, e.g., $\varphi(p) = p - 1$).

The function $\varphi(m)$ has a number of wonderful properties and the study of it opened a new and important chapter in the number theory: That of arithmetic functions in integer argument. The Euler function was the first of them.

In the same years, Euler had experimentally discovered that, for any prime p, a number a can be found, such that $a^k - 1$ is divisible by p for all $k < p - 1$ (or equivalently, that the residues of a^k modulo p are all distinct). For example, $a = 3$ satisfies the statement for $p = 17$. Euler has called such a's *primitive* *roots for p*. In 1769, they had also been already considered by J. H. Lambert (1728–1777). Euler had conjectured that a primitive root exists for any prime p and indicated an outline of the proof. A complete proof of the conjecture was given by A. M. Legendre (1752–1833).

Later, C. F. Gauss (1777–1855) gave another proof. Gauss had also found a way to compute the number of primitive roots of p. Manipulations with primitive roots were for Gauss a good starting point for the construction of a regular 17-gon by ruler and compass; he accomplished this construction when he was only nineteen.

Another series of questions Euler inherited from Fermat were concerned with the study of residues of squares. In elementary problems, it is often useful to remember that the residue of n^2 modulo 3 can never be equal to -1, e.g., one can quickly establish that a certain very large number is not a square. Fermat claimed that -1 is never the residue of a square modulo any prime of the form $4k-1$, but ought to be the residue of a square modulo some prime of the form $4k+1$. Euler had proved Fermat's conjecture in 1747 and then tried to investigate a similar question replacing -1 by an arbitrary integer a. A wonderful conjecture had arisen:

Consider an arithmetic progression $b+4ak$, where a and b are relatively prime, $k=1,2,\dots$. Then either a is not a residue of a square modulo any prime from the progression or, for any such prime, a can serve as a residue.

Euler had proved his conjecture only for $a=3$ but failed to prove it for $a=2$. Later Lagrange proved the conjecture for $a=2,5,7$; some other cases were considered by Legendre who also tackled the proof in the general case but his proof contained essential gaps.

The first complete and extremely difficult proof was given by Gauss (a week after he had constructed a 17-angle). He called the Euler's conjecture a "golden theorem", but nowadays it is called the *quadratic reciprocity law*.

Fermat had also claimed that

any prime of the form $4k+1$ is representable as a sum of two squares, moreover, in a unique way (and primes of the form $4k+3$ can not be represented as sums of two squares).

When Euler proved this statement, he also found that the converse is also true: If a composite number is representable in the form of two squares, then the representation is never unique. He noted that this observation can be used to prove that a certain number is a composite one. Later he discovered that the case of the representation in the form $x^2 + 2y^2$ is similar: If a number is uniquely representable in this form, then it is necessarily prime.

He verified that a similar statement holds for the representations in the form $x^2 + 3y^2$ and $x^2 + 4y^2$. A number a is said to be *convenient* if only primes are uniquely representable in the form $x^2 + ay^2$. All $a < 10$ turn out to be convenient, and each case is not easy to verify.

But how deceptive can partial induction be! The number $a = 11$ is not convenient. This discovery did not close the problem for Euler. He continues to experiment and finds out that the numbers 12, 13, 15, 16, 18, 21, 22, 24 are convenient, but gradually the convenient numbers become more and more scarce and in the first thousand there are only 62 convenient numbers.

Euler tirelessly continues his computations; there are only three convenient numbers in the second thousand: 1320, 1365, 1848. In the third thousand there are no convenient numbers at all, and no more were discovered, although Euler patiently continued his computations up to $a = 10000$. On the ground of these computations Euler conjectured that there were exactly 65 convenient numbers. Gauss reckoned Euler's computations but did not prove his conjecture. Only in 1930s the finiteness of set of convenient numbers was proved, and at present it is known that there are not more than 66 of them.

This Euler's work is comparatively isolated from his most spectacular results but it clearly shows his style. What a persistence, perseverance, intuition and scientific courage were necessary to proceed with the computations until serious grounds for a conjecture that the set of convenient numbers is finite were unearthed! One can only admire the grip Euler should have had to continue the computations far enough to obtain confidence in his conjecture.

As a Fermat's disciple, Euler could not avoid problems connected with the solution of the equation in integers. In 1759, Euler considered the equation

$$x^2 - Dy^2 = 1, \text{ where } D \neq a^2,$$

which he names, not very appropriately, *Pell's equation* although it should have been called the Fermat equation. Euler established a connection of the smallest positive solution of the equation with the decomposition of D into the infinite continuous fraction and had noted that in all the examples considered the fraction is periodic. He conjectured that *all quadratic irrationalities (and only them) can be expanded into a periodic continuous fraction.* The honor to prove this Euler's conjecture fell to Lagrange's lot.

Euler also gave the first proofs of the facts related to Fermat's Last Theorem (*the equation* $x^n + y^n = z^n$ *has no integer solutions for any integer* $n > 2$). In 1738, Euler obtained the proof for $n = 4$, and in 1755, for $n = 3$ having reconstructed Fermat's method of infinite descent.

It is remarkable that apparently he never seriously tried to obtain a proof for an arbitrary n (Gauss also never did that) and was not seduced by Fermat's assuredness. In these cases Euler stands out as Fermat's disciple in the highest sense of the word. He had incorporated Fermat's statements into a thoroughly elaborated multiplicative (i.e., related with divisibility) number theory perceiving infallibly almost all of its main theorems and problems. Proofs of several key statements were left to Euler's successors.

The examples we have given clearly show Euler's scientific style. He had an excellent choice of wonderful problems he could concentrate upon for years if not for the whole of his life, but there was no concrete problem that for Euler had the priority before both the art of creating of the whole picture, and his irresistible desire to move forward. He persistently returned to problems he was unable to solve, skillfully measuring the time he could spare for any particular problem. The difficulty of the problems, the fact that he had often had to abandon his attempts to obtain a rigorous proof, had led Euler to ways of establishing mathematical truths which were different from proofs.

Experiment came into a foreground not only at the preliminary stage of thinking over problem or formulating a conjecture. In Euler's inner value system, a numerical test skillfully carried on a vast material was sometimes equivalent to the very proof of the truth. He spoke about "truths perceived but not proved" and tried to ensure that argumentation of such kind obtained the rights of citizenship in mathematics. To find a rigorous proof remains the most important aim for Euler, but at a certain stage of an investigation he consciously refused a further search for the proof and devoted himself to meticulous development of heuristic considerations.

We have listed several Euler's conjectures proved later. Indeed, the proofs devised for some of them brought glory to the greatest mathematicians. But, apart form conjectures, there were other statements Euler left without proof. Euler considered experiment as an important way of establishing mathematical truth not only in the number theory. For example, instead of giving a proof for his formula for polyhedrons, he had at first only verified it for all polyhedrons which were known to him, and then considered the formula "sufficiently justified".

Fermat's arithmetic does not include the additive number theory. It is not concerned with the problems of divisibility and studies partitions of numbers into sums of other numbers. Some problems of this kind which were connected with primes were discussed in Euler's circle. Several of them were suggested by Euler himself, and he also took an active part in developing the approach to the famous Golfbach's problem (viz., that *any even number* > 2 *is representable as a sum of two primes*), and so on. The problems turned out to be very difficult and no ways to solve them had been found at the time.

In connection with a different series of problems, Euler worked out a method which many years after proved to be helpful in the solution of these problems. This time Euler did not start from deep statements of Fermat but from problems of the comparatively accidental nature which had been communicated to him in 1740 by Ph. Naudé, Jr. (1684–1746) whose name hardly says anything to our contemporaries. These were problems on a number of representations of an integer of the form of the sum of summands of a special form, and these problems are rather combinatorial than number theoretical.

The astounding Euler's idea was to apply infinite series and products to the solution of these problems. Here is an example. Write down the identity

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = 1+x+x^2+x^3+x^4+\ldots$$

To check its verisimilitude, it is convenient to multiply both sides by $1-x$ and check that the non-negative powers of x vanish consequently. If, in the left-hand side, the multiplication is formally performed, then the coefficient of x^k is equal to the number of representations of k as the sum of different powers of 2, and comparison with the right-hand side shows that such a representation is always possible and unique.

It goes without saying that this argument is not rigorous, but Euler was not taken aback. He looked through a great number of different examples bravely applying analytical operations (differentiation, integration) and algebraic operations to infinite sums and products. Euler himself was amazed by the possibility to apply analysis to arithmetic: "*Though we discuss here the nature of integers to which Calculus seems inapplicable, I have, nevertheless, come to the results of my deduction with the help of derivations and other tricks*".

The most difficult arithmetic problems are connected with primes. The very first problem is to construct them. Euler has disproved a method proposed by Fermat (see above). In the framework of another method going back to M. Mercenne (1588–1648) Euler managed to produce a 10-digit prime. Euler also showed that there is no polynomial $P(x)$ whose values at integer points are prime.

Euler was anxious to find how primes are distributed among integers. His idea was to apply analysis to this problem. It marked the beginning of the Analytical Number Theory.

Consider the series

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots. \tag{2}$$

It was not later than 1737, when Euler had found that this sum coincides with the infinite product

$$\zeta(s) := \prod \frac{1}{1 - \frac{1}{p^s}}. \tag{3}$$

where p runs over all primes. The convergence was not discussed, and hence to carry out a plausible argument, the formula for the sum of infinite geometric progression was enough. Euler felt that estimates for $\zeta(s)$ can be reformulated into an information about primes.

But his imagination had lead him too far! He managed only to find a new proof of the theorem on the infinity of the set of primes, known already to Euclid, connecting this fact with the divergence of a harmonic series. Euler discerned one theorem in proof of which Euclid's method can no more compete with his one: *The infinity of the set of primes in arithmetic progression $an + b$, where a and b are mutually prime*. This statement is now known as a theorem by L. P. G. Dirichlet (1805–1859), who proved it using a generalization of Euler's technique.

Passing to partial sums, Euler conjectures that $\ln(\sum\limits_{n<x} \frac{1}{n})$ is close to $\sum\limits_{p<x} \frac{1}{p}$, where the p's are primes. This was the first step toward an asymptotic law of distribution of primes!

A century will pass and the great B. Riemann (1826–1866) will extend $\zeta(x)$ into the complex domain and, as a cornerstone of his theory, will establish the functional equation that relates values of $\zeta(s)$ and $\zeta(1 - s)$ (this Riemann's functional equation was, actually, known to Euler) and the relation with arithmetic will come clear and justified. The function $\zeta(s)$ is called the *Riemann zeta-function* while (3) retains the name of *Euler's product*.

Euler did not pass past any field of his contemporary mathematics, but still, for him, the pupil of Johann I Bernoulli and the scientific grandchild of G. W. Leibniz, Analysis was above all.

It is important to recall the mathematical situation at the beginning of Euler's activity. The whole of XVII century was almost completely open to begin the creation of what is now called the Differential and Integral Calculus. The classics of Analysis were constructing Calculus being sure to obtain algorithmic procedures for solutions of all analytical problems, without exception.

First, everything presaged a happy flourish of the Calculus: Integration produced either elementary functions or newly put into circulation logarithmic, and direct and inverse trigonometric functions. But then evil portents appeared. The "Arberdean's hermit" Gregory (1638–1675) stubbornly insisted that the problem of a computation of the length of an arch of an ellipse or hyperbola leads to integrals which cannot be expressed in elementary functions (elliptic integrals!) and even claimed having a proof of this statement (simultaneously with the transcendence of π) with the help of series.

Practical I. Newton (1643–1727) who also knew of other examples of non-integrabilities (differential binomial for certain values of parameters) did not bother with all this since he did not see here any inhibitions to solve analytical problems with the help of series (e.g., by means of the method of indeterminate coefficients).

For Leibniz, on the contrary, the problem of "closedness" of Calculus seemed crucial (as can be viewed even from his letter to Newton in 1676). His program was to extend the class of elementary functions by higher transcendental functions, so that all analytical problems were solvable. However, nobody could guess which way to go for a realization of this program. For a time being, it seemed that the desired extension of a supply of functions will be obtained by the integration of the rational fractions with denominators of high degree, and for a long time Leibniz could not believe Johann I Bernoulli that there would be no new transcendences.

It is very interesting to trace the dynamics of his estimates of mathematics' future at the turn of the century. It was in 1682 that Leibniz hoped to completely finish the construction of Analysis (and therefore of practically the whole mathematics) in order to put in the next century all strength onto physics (a letter to Huygens). In 1708, in a letter to L'Hospital, he states that "we are no more masters" in the problems of integration, solving differential equations and that in general "analysis of infinitesimals makes only its first steps".

Let us try to consider the opportunities which a young mathematician who had just begun to study analysis in XVIII century might have had. First of all, it was necessary to bring in order what had been done in the previous century, in short, to write textbooks. At the end of the previous century, Johann I Bernoulli and L'Hospital started this work. During the whole of his life, Euler has conscientiously writing a voluminous course of calculus. Not only a systematization of analytical facts was necessary, but also a more reliable foundation, main notions needed to be thought over, and it was necessary to make the main concepts precise. The classics of Analysis assured the high speed of moving forward at the cost of a liberal handling with actual infinity while manipulating with infinitesimals. The picture became clarified no earlier than a hundred years after. It was only recently that creation of the Nonstandard Calculus justified some of old works by Euler.

Though Euler mostly worked in the environment of concrete facts and examples, he took part, which matched his strength, in a process of clarification of main concepts. However, his merits in this direction are, perhaps, less than those of his contemporary — d'Alembert — who had a more humane style of thinking (besides, specific requirements were imposed on d'Alembert by the necessity to write popular articles on Calculus for "Encyclopeadia"). But when a precision of a notion was necessary for a concrete problem, Euler was at his best. So it was in the problem of a string's oscillation that Euler categorically claimed that at the initial moment a string can take any form which can be described by a "free movement of the hand".

Lagrange and Riemann emphasized that it was the moment when arbitrary functions characterized by their graphs rather than by analytical expressions appeared in Analysis. At the same time, d'Alembert, who solved the string's equation one year before Euler, came across unsurmountable difficulties at the attempt to combine analyticity of the initial function with periodicity, and was forced even to prohibit the string to be bent at the initial moment along a parabola.

As to concrete analytical problems, it became very hard to find worthy problems — after the giants of the previous century had systematically looked through the possibilities at hand. And, in fact, in the Analysis itself, in XVIII century, few outstanding results were obtained.

Nevertheless, Euler had found some missed opportunities. One of them concerns the Newton's preferred field of science, the theory of series. Euler systematically considers series as polynomials of infinite power. In particular, he expands them into an infinite product of linear multiples with respect to their roots (with certain modifications as compared with ordinary polynomials since the non-existing highest term should be replaced by the lowest term). For instance, since, for

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

the roots are $k\pi$, we have

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\dots \tag{4}$$

Multiplying formally, simplifying and gathering the coefficient of x^2 we get

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\dots = \frac{\pi^2}{6} \tag{5}$$

i.e., the value of $\zeta(2)$. It was already Jacob I Bernoulli (1654–1705) who dreamed of summing up the series (5). Euler first counted the sum with 7 digits, later he improved his calculation to 15 digits. Understanding the absence of rigor in his arguments he begins to compare this result with the right-hand side. Note that the value of $\zeta(2n)$ for any n is easy to find from eq. (4). The problem of computation of $\zeta(n)$ at odd integers appealed to Euler very much. Trying to solve it he observed the relation between $\zeta(s)$ and $\zeta(1-s)$, but only in the very recent years something about the arithmetic nature of $\zeta(2n+1)$ became known.

Vexed by constant grudges against his arguments, Euler repeatedly tried to justify the expansion of $\sin x$ into infinite product (in particular, with the help of the formula for e^{ix}). Without expecting to obtain complete justification he collects indirect evidence. For example, he obtains by his method a relation

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

which had been already rigorously derived by Gregory and, independently, by Huygens via the expansion of $\arctan x$.

Euler highly estimated this coincidence: "*One finds here great confirmation of our method which might seem insufficiently secure. Therefore, we must not at all doubt in other results derived by this method*".

An often mentioned peculiarity of Euler's publications is that **they contain**

not only results but also invaluable information on the ways to obtain them.

One should not forget though that the very style of Euler's work, who often did not possess any proofs, did not leave him other possibilities to persuade the reader, except to confine the reader in all his heuristic considerations.

To finish the topic on Euler's relation with series, notice that even however rigorous theory of convergence of series and infinite products which appeared fifty years after would not have resolved all Euler's difficulties. Unlike Newton or d'Alembert, he insisted on the necessity to work in analysis not only with convergent but also with divergent series.

Here, too, intuition did not deceive Euler. But he looked so far ahead of his time that he lacked not only the technique for proofs but also a language to speak. This has led to a peculiar situation. Whereas Euler was saved from errors by his unseen intuition, his epigones (in the part concerning divergent series) had obtained plenty of absurdities which defamed Euler's project.

On the border of Analysis of XVII century, variational problems growing from the problems on brachistochrone (the curve of the fastest descent) began to appear and arose general interest. These problems (quest for the curves or functions satisfying some minimum-maximum properties) belonged already to another epoch in Analysis and, according to the modern tradition, the Calculus proper terminates just before they begin, and the Calculus of Variations forms a separate field of mathematics which deals not with functions depending on a point, but with functions depending on a curve, i.e. with functionals.

It was actually Euler, followed by Lagrange, who, starting from a collection of separate tricks for solving isolated variation problems, made Calculus of Variations a branch of mathematics. Euler had a great amount of consideration toward 19-years old Lagrange who communicated in a letter (1755) on essential improvements of Euler's calculus. Euler did not hurry to publish his own achievements based on Lagrange's ideas in order to give the young scientist the opportunity to publish his results first.

Euler received the first variation problem in his youth from Johann I Bernoulli and during his whole life remained faithful to the field where his scientific career began. Perhaps that is why he was so deeply moved by P. L. Maupertius' (1698–1759) idea that variation problems are of fundamental character in natural sciences and completely describe real processes in Nature (1744).

In particular, a real mechanical motion takes its course so that a special quantity — *action* — takes the minimal value. Maupertuis' motivation was to an extent of teleological nature, and the minimality of action was perceived as a corollary of the "wisest use of God's might". Denying not this logic, Euler completed these motivations of Maupertius giving an exact formulation of the principle of the minimal action for the case of central forces.

Maupertius' assertions were the center of a bitter discussion and the generality of his statements enabled people very remout from physics to participate in this discussion. "This discussion on an action, if one will be allowed to say, resembles somewhat some religious heated disputes by the bitterness added to it and by the number of participants who do not understand anything in what they participate", d'Alembert said.

Volter was the most violent, he was in old quarrel with Maupertius and issued a satirical pamphlet "A diatribe on doctor Akakia". In the long run, Maupertius was morally defeated, but Volter also smeared Euler, irreconcilable counsel for Maupertuis' defense. Euler can be recognized without fail in Volter's scientist who tries to gain his glory among European mathematicians by "producing on the paper maximum of computations". It is said about a scientist "who computes on no less than 60 pages instead of trying to think and waste no more than 10 lines, who computes for three days and three nights in a fear to spend a quarter of an hour to invent a right way".

That is how the image of a genius human computer was transformed in Volter's head. Even having in mind the polemical nature of the situation, the cock-sure "nifties" are improper for the great philosopher and writer.

Euler had been often reproached, and still is, for overestimating the confused arguments of Maupertuis and for the fact that he almost deliberately stressed the secondary nature of his own works. It was even hinted that practical Euler tried to please the all-powerful president of the Berlin Academy of Sciences.

But it seems to me that to Euler it was a natural relation to the work of Maupertuis: Euler could value pioneering works and used to have a lot of understanding in what an imperfect shape their ideas might be presented. Maupertuis used to say out precisely what Euler might naturally say himself.

Euler always sought for more secure foundation of mechanics than Newton's laws which he was not prepared to take for initial axioms. But it seemed that his fate did not allow him to guess that the necessary principle might be obtained from his favorite variational calculus.

Mathematicians of XVIII century remembered Leibniz's dream to analyze high transcendental functions. However, a sober estimate shows that at that time there were no regular ways to deal with this problem. Several examples of functions appeared in works of different mathematicians. But nowadays we clearly see that it was the problem for XIX century and, simultaneously, that Euler, being led by his sixth sense, guessed and listed, practically without gaps, all special functions which will later constitute the object of the higher analysis.

We have already discussed the Euler integrals and zeta-functions. To them we can add the Bessel functions, several types of theta-functions, hypergeometric Gauss's function which turns at certain values of parameters into most of special functions encountered in mathematical physics.

Finally, Euler has made the most important steps in the theory of elliptic integrals, addition theorem including. These results were a starting point for Legendre and Gauss, Abel and Jacobi. It became a custom that if a new natural class of functions arises, one must look through Euler's list for it.

In the recent years in the most different problems of the number theory, algebra, topology and geometry, the *dilogarithm*

$$\mathrm{Li}_2(z) := \sum_{n\geq 1} \frac{z^n}{n^2}$$

mystically appears. It turned out that Euler knew of wonderful properties of this function, too, in particular, about the addition theorems.

The most important technical device, which Euler lacked, is the extension of special functions into the complex domain. But Euler had already performed the first steps in the construction of the complex analysis. Like d'Alembert (though in the relation with problems of hydromechanics) he had considered the Cauchy-Riemann equations which define analytical functions of a complex variable; he employed complex substitutions for computing real integrals, and, in his latest years, calculated real integrals via integrals of complex functions closely reaching the Cauchy theory of contour integration in the complex domain.

It is impossible to separate the search for special functions from the task of distinguishing the important classes of differential equations. Already at that time, there was nobody who would doubt the impossibility to integrate explicitly an arbitrary differential equation. Euler actively participated in discovering the equations that arise from physics. He considers a series of equations in the relation with problems of hydromechanics, oscillations of strings and membranes, spreading of sound: Here we find the Laplace equation and some versions of the wave equation, and so on. It was natural for Euler to look at physics from the analytical viewpoint. He tried to reduce physical problems to the solution of certain differential equations. In mechanics, he was the first to use the coordinate method; he passed from the Newton's geometric language to the analytic one (Lagrange had even developed Euler's ideas in the book "Analytical mechanics" which contained no pictures).

Summing up Euler's activity in Calculus, let us emphasize that Euler preferred analytic methods for solutions of both strictly mathematical and applied problems. But, for Euler, analysis was never a goal in itself. One can recall that he assiduously searched (unlike d'Alembert) for a purely algebraic proof of the main theorem of algebra (on the existence of a complex root of any algebraic equation). He could not find it and G. Frobenius (1849–1917) remarked with pity that these algebraic considerations of Euler were not given their due and many of them are unjustly attributed to Gauss.

We were able to retell only some of directions in Euler's scientific activity. He was uncompromising in his scientific tactics: He preferred wide frontal attacks on scientific mysteries without concentrating his strength on isolated problems. For this choice he had to pay considerably. We have seen already lots of examples when he left to his successors to accomplish ("to cover up his traces"), to prove theorems he had guessed. It is also possible to give examples where he was outridden by his contemporaries whom greater concentration in the study was inherent. Euler was a calculator by God's mercy (as Arago said, "he calculated as one breathes").

At that time, there was no more natural field to apply his abilities for computations than celestial mechanics. Its main problem inherited by XVIII century was to match the universal gravitation law with real observations. On the level of pair-wise interactions (the problem of two bodies: the Sun and a planet, the Earth and the Moon) the coincidence was astounding. But some very real deviations remained which it was natural to explain by the influence of third bodies (the Sun on the Moon, mutual attraction of large planets). There was another possibility which was seriously discussed: To correct the universal gravitation law, e.g., modify somewhat the exponent 2 in the formula

$$F = G\frac{m_1 m_2}{r^2}.$$

To explain this discrepancy between the experiment and theory was not an easy task. Euler is the founder of computational methods in celestial mechanics; he was indefatigable in concrete computations, but still: Which "awards" did he receive?

One of the most difficult problems was to explain the periodicity with the period of 9 years of the motion of the perigee of the orbit of the Moon. Perturbations up to certain terms being taken into account stubbornly yielded a period of 18 years until in 1749 Clairaut showed that perturbation terms of higher order being taken into account yield the right answer. In his reference to the Clairaut's work in connection with the presentation to the prize of the St. Petersburg Academy of Sciences Euler wrote "*in this question Mr. Clairaut did not have, perhaps, a stronger opponent than me... though I was in this question a predecessor of Mr. Clairaut, I lacked enough patience to launch into so extensive computations*". Clairaut possessed one more, no less effective, result: The delay of Galley's comet at its next return. Had Euler got any results as brilliant and easy to understand to a *wide* audience? Perhaps there were none.

However, we can recall his theory of the Moon based on which I. T. Mayer (1723– 1762) comprised lunar tables so precisely that they were accepted for measurements of the longitude aboard ships (this was reflected in awarding Mayer and Euler the premiums of the English parliament in £3000 and £300, respectively).

Euler tried to grasp each opportunity to look ahead, to foresee a farthest future of his science. He picks up each occasional problem in hope that it can lead him in an entirely new direction. We can recall the popular problem on Königsberg's 7 bridges which are impossible to encompass without stepping twice on one of them.

In March 1736, he wrote: *This question, though banal, seemed to me worthy of exploring, since to solve it, neither geometry, nor algebra, nor combinatorial skill were sufficient. Therefore, it came to my mind, does it not, accidentally, belong to the "geometry of position" investigated at his time by Leibniz.* (In a letter to Marioni.)

Indeed, Leibniz has left several cryptic asides on a mysterious geometry "which manifests to us in position like algebra in quantities". (A letter to Huygens, 1679.) Euler vainly tries to find out details about this "geometry of position". In a letter to Ehler, 1736, Euler asks why "such questions of little relation to mathematics are sooner solved by mathematicians rather than by somebody else".

Euler saw the situation in the proper perspective: Methods of argumentation which he suggested belong to foundations of topology, traditionally taken as an embodying of Leibniz's "geometry of position".

Euler is one of the greatest mathematicians of all times. The style of studying mathematics inherent to him was ineffable. He knowingly chose his fate preferring the search for new ways and trials rather than penetrating along them as far as possible (although he walked along them considerably) and rather than to completing solutions of concrete problems (especially if he had no doubt in the answers). It is astounding even now how far ahead he could foresee the development of mathematics. Its future arose before him in a row of remarkable examples which served as a starting point to mathematicians who followed him to recreate the whole picture.

Jacobi wished to guess the mystery of everlasting success of Euler's books: Why did he (Jacobi) read them without granting a discount because of their age (70 years after the author's death). Jacobi saw the secret in the main contents of Euler's books — a collection of wonderful examples not subject to aging.

Only those can truly estimate Euler who were taught by his works, used his prophetic ideas and undertook the hardest burden to "cover up Euler's traces", supporting validity of his guesses, who discovered what was not Euler's fate to see.