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STATISTICAL APPROACH TO NON-LINEAR CONTROL SYSTEMS

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Abstract. Here we present the statistical approach to non-linear control system and time-correlation function formalism to investigate correlation properties of a control system with the statistical type of behavior (in the form of speed gradient method).

Key words : statistical control
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Introduction: Physics Background of the Mathematics Formulation

Application of control theory to dynamical non-linear systems in the form of speed gradient method is stimulated by the practical problems of modern physics and technology, how to choose appropriate parameters of the optical field to achieve the most efficient focusing of the cooled atomic beam in periodical potential wells of laser standing wave?

Generally speaking, we want to describe the ensemble of classical or quantum particles, when for every atom from the ensemble we have the second order ODE based on the Newton or Schröedinger equation correspondingly. The coefficients of this equation depend on the "outer" field (say, in the case of nanolithography focusing, the laser field consisted actually of two fields: basic and additional modulated). Because of the field modulation the coefficients are changed and they are time-dependent, it influences on the particle dynamics, and we get the opportunity to realize a feedback, i.e. to express the process in terms of control theory and to apply control methods to the dynamical differential equations [1].

Standard speed gradient method, however, is not very productive in nanotechnology and in the similar models. From one side, it allows achievement of the selected level of energy (the eigenfunction of Hamiltonian for dynamical differential equation). But for the practical purposes of nanolithography we need to prepare an atomic beam with the very narrow spatial distribution. Thus, it is logical to include the parameters of the desired distribution in the goal function of the control process. For this purpose we have to reformulate the mathematical task and introduce the new type of controlled systems. We demand the achievement of the narrow spatial distribution of the dynamical particles as the main goal of the control. Additionally we investigate the statistical properties of the particle dynamics but not the behavior of the single particle. Statistical approach is very natural if we discuss the dynamical properties of a quantum system, but this problem was not formulated in the frame of classical control theory of real systems. At present only separate cases of atomic beam dynamics are investigated [1,2].

Thus, in this paper we consider the statistical description of controlled system:
1. We realize spatial control in the place of energy level control, here we describe it in the form of speed gradient approach, but it can be easily reformulated in the frame of other control scheme;
2. We introduce the statistical control for the ensemble of particles in the place of "individual" control for the dynamics of a separate particle. These particles interact only with an external field, consisted of basic field and additional modulated field to establish some form of control in the system.

General Principles of Statistical Control

Let's discuss statistical description of a system with N classical particles [4]. For statistical description of an ensemble of N identical classical or quantum particles we need to define the density function ρ with the condition:

$$dP = \rho dx$$

where dP is the probability to find a particle in the small volume dx of the state vector R^n-space. Sure, the density function has a normalization:

$$\int \rho(x)dx = 1 \tag{1}$$

(with integration over all possible x), i.e. the probability to find the particle somewhere in the state vector space is equal to 1. The density function ρ represents all N particles of the ensemble. The particles are completely equal, the only difference is in the initial conditions. The average of some space-dependent $A(x)$ should be defined in this case as:

$$\bar{A}(x) = \frac{\int A(x)\rho(x)dx}{\int \rho(x)dx} = \int A(x)\rho(x)dx \tag{2}$$

because of normalization (1).

For this discrete system of N particles we can involve:

$$\rho(x,t) = \frac{1}{N}\sum_{\alpha=1}^{N} \delta_n \left(x(t) - x^{(\alpha)} \right) \tag{3}$$

$\alpha = 1, \ldots.., N$ and $x^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, ..., x_n^{(\alpha)})^T$ is the state vector for the αth particle. The Dirac n-dimensional delta function $\delta_n(x)$ is defined as multiplication of one dimensional delta functions $\delta(x_\beta)$:

$$\delta_n(x) = \prod_{\beta=1}^{n} \delta(x_\beta) \tag{4}$$

Every particle with corresponding value $x^\alpha(t)$ is considered under the rule of dynamical equation

$$\dot{x} = F(x, u, t)$$

with the same function F. With ρ_0 we denote the initial distribution function, and with ρ_* – the desired one. The goal function in such a control can be chosen as

$$Q = (\rho - \rho_*)^2 \ .$$

For the purpose of correct formulation for control mechanism in our non-linear system we will replace the discrete function (3) with a smooth differentiable model $\rho(x)$. We will use the one dimensional Dirac function model:

$$\delta_1(z) = \frac{1}{\epsilon\sqrt{\pi}} e^{-z^2/\epsilon^2} \ , \tag{5}$$

where z is from R^1, $\epsilon \to +0$.

It is sure that these models save the normalization:

$$\int_{-\infty}^{\infty} \delta_1(z) = 1$$

Then:

$$\rho(x) = \frac{1}{N}\sum_{\alpha=1}^{N}\prod_{\beta=1}^{n} \delta_1 \left(x_\beta - x_\beta^{(\alpha)} \right) \tag{6}$$

General Principles of the Time-Correlation Formalism

In our description of the time-correlation function formalism we follow the standard statistical mechanics concept [5]. Here we shall do this in the classical limit, but there is a quantum-statistical analog.

Let $x(t)$ denote all the state space coordinates and let $A(x,t)$ be some function of them. We can define a classical time-correlation function of $A(t)$ by

$$C_{AA} =< A(0)A(t) >= \int dx A(x,0)A(x,t)\rho(x) \ , \tag{7}$$

where ρ is the density function of our ensemble.

We can also write time-correlation functions C_{AB} of two state space functions, $A(x,t)$ and $B(x,t)$, i.e. $< A(0)B(t) >$. Additionally we introduce some generalized susceptibility $\psi(\omega)$ in the form:

$$\psi(\omega) = \int_0^\infty dt \ e^{-i\omega t} < A(0)B(t) > \tag{8}$$

The zero-frequency limit of such expression in statistical mechanics has a name of transport coefficient. When $A = B$, the correlation function is often called an autocorrelation function.

Since the density function has been chosen in the form of the delta function sum (6), we can take the integral in Eq. (7) and rewrite it in the form:

$$C_{AA} = \frac{1}{N} \sum_{\alpha=1}^{N} A(x^{(\alpha)}(0),0) A(x^{(\alpha)}(t),t)$$

and

$$C_{AB} = \frac{1}{N} \sum_{\alpha=1}^{N} A(x^{(\alpha)}(0),0) B(x^{(\alpha)}(t),t)$$

These time-correlation functions play a similar role in non-equilibrium statistical mechanics that the partition functions play in equilibrium statistical mechanics (for more details see [5]). Moreover, it is quite typical in statistical description to apply a time-dependent perturbation to a system and assume that this perturbation induces a time-dependent response which is linearly related to the perturbation. For this reason, the time-correlation function approach is often called linear response theory.

Further Reading

Statistical control is quite a new area, still it is waiting for young specialists to develop this approach, and you can hardly find some popular literature on it. In [4] the basic ideas described shortly in the present article has been presented in extended way with some examples.

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