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Part 1

On the Hardy Hilbert Space $H^2(E)$ and the Model Space $\mathcal{K}_\Theta$

by

Rewayat Khan

(rewayat.khan@gmail.com)

Abstract

Our setting is, $\mathbb{D}$ the open unit disc, $\mathbb{T}$ the unit circle in $\mathbb{C}$ and E will represent the Hilbert space ($dimE < \infty$). The Hardy Hilbert space $H^2(E)$ is the space of E- valued analytic functions on $\mathbb{D}$ with expansion $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, such that

$$\sum_{n=0}^{\infty} \|a_n\|^2 < \infty.$$

where $a_n \in E$. We will emphasize the manner in which $H^2(E)$ sits as a subspace inside $L^2(E)$. In this talk we will also discuss the *model space* denoted by $\mathcal{K}_\Theta$ which corresponds to an inner-function (or unitary operator in the generalized sense) $\Theta(z)$ and is an S^*-invariant subspace of $H^2(E)$ where S^* is the backward shift operator on $H^2(E)$. We will discuss in more detail the action of the linear operator S_Θ and S^*_Θ on the *model space* $\mathcal{K}_\Theta$ and the orthogonal decomposition of $\mathcal{K}_\Theta$ in terms of subspaces $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ and its orthogonal complements, where

$$\mathcal{D} = \{(I - \Theta(z)\Theta(0)^*)\zeta : \zeta \in E\} = P_{\mathcal{K}_\Theta}\zeta$$

and

$$\widetilde{\mathcal{D}} = \{\frac{1}{z}(\Theta(z) - \Theta(0))\zeta : \zeta \in E\}.$$

$P_{\mathcal{K}_\Theta}$ is the orthogonal projection on $\mathcal{K}_\Theta$ and ζ denote the constant functions in E.
From the inner function $\Theta(z)$ one can construct the function $\widetilde{\Theta}(z) = \Theta(\overline{z})^*$. Then $\widetilde{\Theta}(z)$ is also inner and the *model space* associated to $\widetilde{\Theta}(z)$ is $\mathcal{K}_{\widetilde{\Theta}}$. We will emphasize the connection between the *model spaces* $\mathcal{K}_\Theta$ and $\mathcal{K}_{\widetilde{\Theta}}$ through a unitary operator τ defined on $L^2(E)$.
For $f \in \mathcal{K}_\Theta$, the orthogonal projection on $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ is denoted by $P_{\mathcal{D}}$ and $P_{\widetilde{\mathcal{D}}}$ respectively which are defined by

$$P_{\mathcal{D}}f = (I - \Theta(z)\Theta(0)^*)(I - \Theta(0)\Theta(0)^*)^{-1}f(0),$$
$$P_{\widetilde{\mathcal{D}}}f = \frac{1}{z}(\Theta(z) - \Theta(0))(I - \widetilde{\Theta}(0)\Theta(0))^{-1}(\tau f)(0).$$

We will also investigate the formula for $I - S_\Theta S^*_\Theta$ and $I - S^*_\Theta S_\Theta$ like in paper of D.Sarason for $E = \mathbb{C}$.