# Entanglement in QFT and Holography

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## Abstract

We present several methods to compute the spatial entanglement entropy of a QFT state and illustrate these methods with simple examples. We also discuss the connection between the entanglement entropy and conformal anomalies.

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**CONTENTS**

Entanglement is one of the most fundamental properties of quantum mechanics. Entanglement is an interesting and useful concept in many different fields of physics, such as quantum information, condensed matter physics, quantum field theory, and quantum gravity. In these notes, our focus is merely on to study the entanglement between spatial subregions in a state of a quantum field theory. We encourage the readers to consult [1] for more details.

## I. PRELIMINARIES

Here we present some definitions and formalisms that we will use to discuss the spatial entanglement in quantum field theory in the rest of these notes.

### A. Bipartite system and reduced state

Suppose we have a bipartite system (system $A$ and system $B$). The Hilbert space of the full system can be factorized as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \,. \tag{1}$$

Let the full bipartite system is in a pure state $|\psi\rangle$. We can represent this state in terms of a density matrix

$$\rho = |\psi\rangle\langle\psi| \,. \tag{2}$$

However, we do not need the full density matrix, $\rho$, to describe the state of one of the subsystems, say system $A$. Consider an operator that only acts non-trivially on $\mathcal{H}_A$, such as $\mathcal{O} = \mathcal{O}_A \times \mathbf{1}_B$. The expectation value of this operator is then

$$\mathrm{tr}\,(\mathcal{O}\rho) = \mathrm{tr}_A\,(\mathcal{O}_A\rho_A) \;, \tag{3}$$

where

$$\rho_A = \mathrm{tr}_B\,\rho \,, \tag{4}$$

is called the reduced state of system $A$. This reduced state is important in the cases where we only have access to system $A$. The reduced state, $\rho_B$, for system $B$ can be defined in a similar way.

### B. Entanglement, entanglement entropy, and Renyi entropy

A pure state of a bipartite system is said to be a *product* state if it can be factorized as

$$|\psi\rangle = |\phi_A\rangle \otimes |\chi_B\rangle \,. \tag{5}$$

On the other hand, states of a bipartite system that cannot be factorized in this way are called *entangled* state. Note that a general pure entangled state can be expressed as

$$|\psi\rangle = \sum_{i_A=1}^{N_A} \sum_{i_B=1}^{N_B} c_{i_A,i_B} \, |i_A\rangle \otimes |i_B\rangle \,, \tag{6}$$

where $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are orthonormal basis for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Applying Eq. (4) for this entangled state yields

$$\rho_A = \sum_{i_A=1}^{N_A} \sum_{j_A=1}^{N_A} \left( \sum_{i_B=1}^{N_B} c^*_{i_A,i_B} c_{j_A,i_B} \right) |j_A\rangle\langle i_A| \,, \tag{7}$$

which is a mixed state. This gives us an alternative definition of the entangled state: A pure state is entangled if the reduced state of any subsystem is a mixed state.

A useful measure of the entanglement of a pure state is the entanglement entropy. The entanglement entropy of a subsystem $A$ is defined as the von Neumann entropy of the reduced state $\rho_A$. That is,

$$S_A = -\mathrm{tr}_A \, \rho_A \log \rho_A \,. \tag{8}$$

Note that $S_A$ vanishes if the reduced state $\rho_A$ is a pure state, which would be the case if the pure state $|\psi_{AB}\rangle$ is unentangled.

Another quantity closely related to the entanglement entropy, called $n^{\mathrm{th}}$ Rényi entropy, is defined as

$$S_A^{(n)} = \frac{1}{1-n} \log \, \mathrm{tr}_A \, \rho_A^n \tag{9}$$

for integer $n \geq 2$. Note that this quantity also vanishes if $\rho_A$ is a pure state. Denoting the $i^{\mathrm{th}}$ eigenvalue of $\rho_A$ by $\lambda_i$, we can write $S_A^{(n)}$ as

$$S_A^{(n)} = \sum_i \lambda_i^n \,. \tag{10}$$

Since $0 < \lambda_i < 1$, this sum is convergent and analytic for $\mathrm{Re}(n) > 1$. Therefore, $S_A^{(n)}$ can be analytically continued for non-integer $n$, and the limit $n \to 1$ reduces to the entanglement

entropy. That is,

$$S_A = \lim_{n\to 1} S_A^{(n)}\,, \tag{11}$$

$$= -\left(\frac{\partial}{\partial_n}\log \mathrm{tr}_A \rho_A^n\right)\Big|_{n=1}\,. \tag{12}$$

This analytic continuation is unique for a finite dimensional Hilbert space [1]. To show this, first note that

$$\left|\mathrm{tr}\rho_A^n\right| = \left|\sum_i \lambda_i^n\right| \leq 1 \qquad \text{for} \qquad \mathrm{Re}(n) > 1\,. \tag{13}$$

Now let's assume that there are two possible analytic continuation of $S_A^{(n)}$ and let's call them $F_1(z)$ and $F_2(z)$. We define the difference of these functions as

$$G(z) \;=\; F_1(z) - F_2(z)\,, \tag{14}$$

where $G(z)$ is an analytic function in the region $\mathrm{Re}(z) > 1$ and it must vanish when $z$ is a non-negative integer. Furthermore, Eq. (13) ensures that $G(z)$ is a bounded function. Now, Carlson's theorem states that an analytic function which is bounded and vanishes at non-negative integers must be identically zero. Hence, there is a unique analytic continuation of $S_A^{(n)}$ to non-integer $n$.

**Exercise 1** *Verify Eq.* (11).

### C. Modular Hamiltonian

Since a density matrix is positive and Hermitian operator with eigenvalues $\lambda \leq 1$, we can write any density matrix as

$$\rho \equiv\; e^{-K}\,, \tag{15}$$

where $K$ is a positive and Hermitian operator, and it is called the *modular Hamiltonian*.

Now, suppose we have a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is a pure state $|\Psi\rangle$. The entanglement entropy of subsystem $A$ is given by

$$S_A = -\,\mathrm{tr}_A\, \rho_A \log \rho_A\,, \tag{16}$$

$$= \mathrm{tr}_A\; \rho_A K_A\,, \tag{17}$$

$$= \mathrm{tr}_A\left[\left(\mathrm{tr}_B|\Psi\rangle\langle\Psi|\right) K_A\right]. \tag{18}$$

Since $K_A$ acts trivially on $\mathcal{H}_B$, we can simplify the last result as

$$S_A = \mathrm{tr}_{A\cup B}\left[|\Psi\rangle\langle\Psi|\, K_A\right], \tag{19}$$

$$= \langle\Psi|K_A|\Psi\rangle\,. \tag{20}$$

This formula tells us that the entanglement entropy can be thought of as an expectation value of a *state-dependent* Hermitian operator. This state-dependence ensures that the entanglement entropy is a non-linear function of the state.

We will use Eq. (20) in Sec. (IV) to derive a perturbative formula for entanglement entropy.

### D. Some properties of the entanglement entropy

Here we list some of the interesting properties of the entanglement entropy.

1. In a pure state of a bipartite system,

$$S_A = S_B\,. \tag{21}$$

This can be seen by noting that a general pure state in Eq. (6) can be written in the *Schmidt decomposition* form

$$|\psi\rangle = \sum_{i=1}^{N_{AB}} \lambda_i\, |u_i\rangle_A \otimes |u_i\rangle_B \tag{22}$$

where $N_{AB} = \min\{N_A, N_B\}$, and where $\{|u_i\rangle_A\}$ and $\{|u_i\rangle_B\}$ are orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

2. The entanglement entropy of two combined systems is lesser than or equal to the sum of the entanglement entropy of individual systems. This is called the *subadditivity* of entanglement entropy,

$$S_{A\cup B} \leq S_A + S_B\,. \tag{23}$$

This inequality implies that the *mutual information* of systems $A$ and $B$,

$$I(A:B) \equiv S_A + S_B - S_{A\cup B}\,, \tag{24}$$

is non-negative.

3. A stronger version of subadditivity is called the *strong subadditivity*

$$S_{A\cup B} + S_{A\cap B} \leq S_A + S_B \,. \tag{25}$$

This inequality can also be written in terms of three subsystems:

$$S_{A\cup B\cup C} + S_B \leq S_{A\cup B} + S_{B\cup C} \,. \tag{26}$$

This form of writing the subadditivity is illuminating as it implies that the system $B$ cannot be purified by both systems $A$ and $C$. This is called the *monogamy of entanglement*.

The subadditivity and strong subadditivity are not easy to prove for a general quantum system. However, we will prove these inequalities in the context of the AdS-CFT in Sec. (V B).

**Exercise 2** *Show that any pure state of the form Eq.* (6) *can be written in the form of Eq.* (22)*. Hint: Perform the singular-value decomposition of the matrix,* $c_{i_A,i_B}$*, in Eq.* (6)*.*

## II. ENTANGLEMENT ENTROPY IN QFT

The Hilbert space of a quantum field theory 'lives' on a Cauchy slice. In these notes, we consider a QFT on a $d$-dimensional Minkowski space, $R^{1,d-1}$. In this case, the Cauchy slice is an equal-time slice which we denote by $\Sigma_t$. We introduce the coordinate system $(t, \mathbf{x})$ where $t$ is a time coordinate and $\mathbf{x}$ are the spatial coordinates on $\Sigma_t$.

In the case of a scalar field theory, which we will focus on in this section, the Hilbert space has a tensor product structure,

$$\mathcal{H} = \bigotimes_{\mathbf{x}} \mathcal{H}_{\mathbf{x}} \,, \tag{27}$$

where $\mathcal{H}_x$ is the Hilbert space of the degrees of freedom at point $\mathbf{x}$ on $\Sigma_t$. To see this, note that a scalar field theory can be thought of a collection of coupled harmonic oscillators at every point in space. Note that this tensor produce structure is not valid for gauge theories due to the existence of non-local operators, such as Wilson loop, in the theory. We reiterate that we do not consider gauge theories here, and we encourage interested readers to consult [2, 3] for discussion of entanglement in gauge theories.

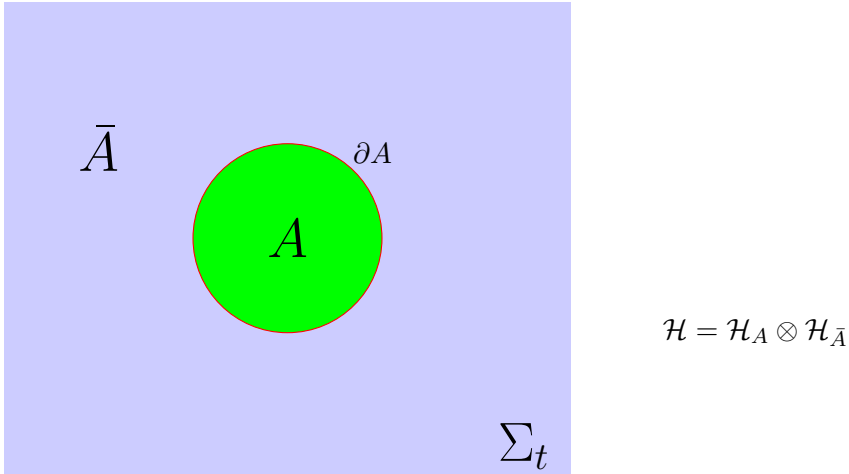


FIG. 1. A codimension-2 entangling surface, $\partial A$, splits a Cauchy slice, $\Sigma_t$, into two regions: $A$ and $\bar{A}$. This provides a bipartite factorization of the Hilbert space: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$.

We can write the Hilbert space in Eq. (27) as a tensor product of two Hilbert spaces. We take a codimension-2 spacelike surface that splits a Cauchy slice, $\Sigma_t$, into two regions. We denote one of these region by $A$, the other region by $\bar{A}$, and the codimension-2 splitting surface, called *entangling surface*, by $\partial A$. See Fig. (1). Now we define $\mathcal{H}_A$ and $\mathcal{H}_{\bar{A}}$ to be the Hilbert spaces of the degrees of freedom in the region $A$ and $\bar{A}$ respectively. That is,

$$\mathcal{H}_A = \bigotimes_{x \in A} \mathcal{H}_x \,, \qquad \mathcal{H}_{\bar{A}} = \bigotimes_{x \in \bar{A}} \mathcal{H}_x \,. \tag{28}$$

With these definitions, Eq. (27) reduces to

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}} \,. \tag{29}$$

Our goal in this section is to show that a vacuum state of a QFT has spatial entanglement. To do this, we calculate the entanglement entropy of a reduced state of region $A$ in a simple example in Sec. (II B). We find that the entanglement entropy for region $A$ is not only non-zero, it is also UV divergent. We discuss the origin and structure of these UV divergences in Sec. (II D).

Finding the reduced density matrix and the entanglement entropy for region $A$ (or $\bar{A}$) is very difficult for an arbitrary entangling surface, $\partial A$. For simplicity, we take the entangling surface to be a codimension-2 infinite plane. To facilitate the discussion, we choose our coordinates $\mathbf{x} = \{x_\perp, \mathbf{x}_\parallel\}$ where $x_\perp$ and $\mathbf{x}_\parallel$ are the coordinates perpendicular and parallel

to the entangling surface respectively. More precisely, the entangling surface is given by

$$\partial A : \; x_\perp = 0 \,, \tag{30}$$

and the region $A$ and $\bar{A}$ are half-spaces,

$$A : \, x_\perp > 0 \,, \qquad \bar{A} : \, x_\perp < 0 \,. \tag{31}$$

Note that the region $A$ is precisely the region that is accessible to an observer with constant acceleration (also known as the Rindler observer) in $x_\perp$ direction. Therefore, we already know that the reduced state of region $A$ is a thermal state with respect to a boost operator. That is, [4, 5]

$$\rho_A = \frac{1}{Z_A} \, e^{-2\pi H_A} \,, \tag{32}$$

where $Z_A = \mathrm{tr}_A e^{-2\pi H_A}$ is the normalization constant, and $H_A$ is the boost operator

$$H_A = \int d^{d-2}x_\parallel \int_0^\infty dx_\perp \, x_\perp \, T_{00}(t=0, x_\perp, \mathbf{x}_\parallel) \,. \tag{33}$$

Since the reduced state in Eq. (32) is a mixed state, we deduce that the vacuum of a QFT is entangled. In the next subsection, we discuss how to represent the density matrix of region $A$ as a path integral.

### A. Path integral representation of a density matrix

Recall that the vacuum state of a Hamiltonian, $H$, up to a normalization constant can be written as

$$|0\rangle \sim \lim_{\beta\to\infty} \, e^{-\beta H} \, |\chi\rangle \,, \tag{34}$$

where $|\chi\rangle$ is a typical (or generic) state. We need to choose a basis to write the wave function of this state. The convenient choice of a basis is the eigenbasis of our scalar field $\Phi(\mathbf{x})$. We denote the eigenstates of $\Phi(\mathbf{x})$ by $|\phi\rangle$ and the corresponding eigenvalues by $\phi(\mathbf{x})$. That is,

$$\Phi(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle \,. \tag{35}$$

The wave function of the vacuum state in this basis is

$$\Psi_0[\phi(\mathbf{x})] = \langle\phi|0\rangle \,, \tag{36}$$

$$\sim \lim_{\beta\to\infty} \langle\phi|e^{-\beta H} \, |\chi\rangle \,. \tag{37}$$

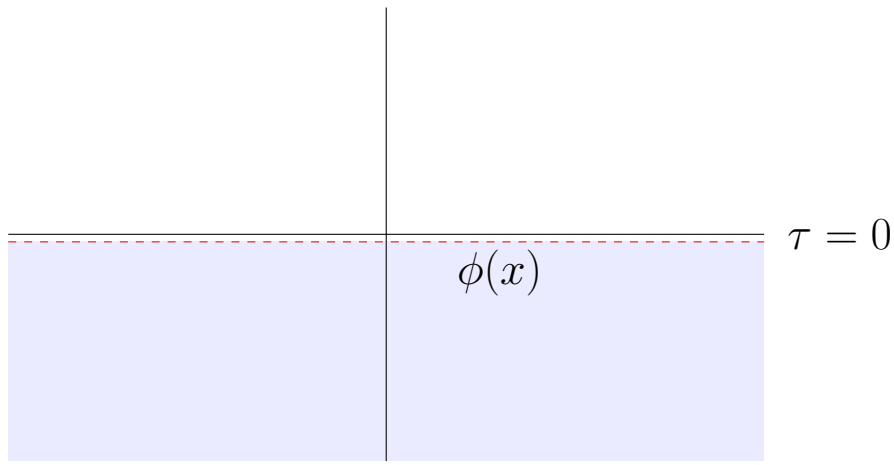


FIG. 2. Pictorial representation of Eq. (38). To get the vacuum state wavefunction, we path integrate over the lower half plane (shaded region) with boundary condition at $\tau = 0$.

The matrix element in the last equation can be represented in terms of a Euclidean path integral over the *lower half plane* . That is, we perform the path integral in the $\tau \leq 0$ region of the Euclidean space, where $\tau$ is the Euclidean time.

$$\Psi_0[\phi(\mathbf{x})] \sim \int_{\tau \leq 0} D\Phi \quad e^{-I[\Phi]} \quad \delta\left(\Phi(\tau = 0, \mathbf{x}) - \phi(\mathbf{x})\right) , \tag{38}$$

where $I[\Phi]$ is the Euclidean action of our theory. See Fig. (2) for a pictorial representation of this path integral.

Similarly, the adjoint of the vacuum state can be written as

$$\Psi_0^*[\phi(\mathbf{x})] = \langle 0|\phi\rangle , \tag{39}$$

$$\sim \lim_{\beta \to \infty} \langle \chi| e^{-\beta H} |\phi\rangle , \tag{40}$$

which can be written as a path integral over the *upper half plane* ($\tau \geq 0$ region of the Euclidean space). That is,

$$\Psi_0^*[\phi(\mathbf{x})] \sim \int_{\tau \geq 0} D\Phi \quad e^{-I[\Phi]} \quad \delta\left(\Phi(\tau = 0, \mathbf{x}) - \phi(\mathbf{x})\right) , \tag{41}$$

Combining Eq. (38) and Eq. (41), we deduce the following path integral expression for the matrix element of the vacuum density matrix

$$\langle \phi_-|\rho|\phi_+\rangle = \langle \phi_-|0\rangle\langle 0|\phi_+\rangle , \tag{42}$$

$$= \frac{1}{Z_1} \int_{R^d} D\Phi \quad e^{-I[\Phi]}\, \delta\left(\Phi(\tau = 0^-, \mathbf{x}) - \phi_-(\mathbf{x})\right) \delta\left(\Phi(\tau = 0^+, \mathbf{x}) - \phi_+(\mathbf{x})\right) , \tag{43}$$

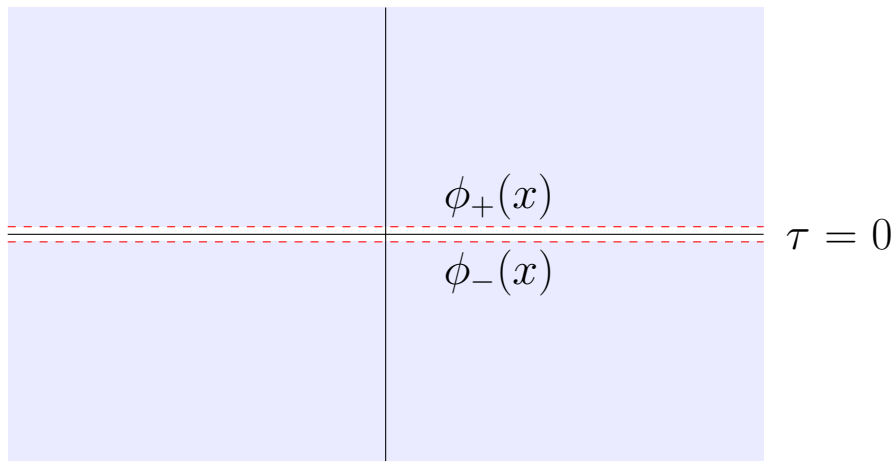


FIG. 3. Pictorial representation of Eq. (43). Path integratal over the full Euclidean space with boundary conditions imposed at $\tau = 0^{\pm}$ gives us the matrix elements of the vacuum denisty matrix.

where the normalization constant, $Z_1$, is the standard partition function of our theory. The pictorial representation of this path integral is given in Fig. (3).

Next, we want to find the reduced density state for region $A$. From Eq. (4), we know that we have to perform the partial trace over the Hilbert space of region $\bar{A}$, $\mathcal{H}_{\bar{A}}$. To perform this analysis, we decompose $\phi(\mathbf{x})$ as

$$\phi(x) = \phi^A(\mathbf{x}) + \phi^{\bar{A}}(\mathbf{x}) \,, \tag{44}$$

where $\phi^A(\mathbf{x})$ only has support in the region $A$ whereas $\phi^{\bar{A}}(\mathbf{x})$ only has support in the region $\bar{A}$. Now the reduced state for region $A$ can be determined by integrating over all $\phi^{\bar{A}}$. That is,

$$\langle \phi^A_- | \rho_A | \phi^A_+ \rangle = \int D\phi^{\bar{A}} \quad \langle \phi^{\bar{A}}; \phi^A_- | \rho | \phi^{\bar{A}}; \phi^A_+ \rangle \,, \tag{45}$$

$$= \frac{1}{Z_1} \int_{R^d} D\Phi \quad e^{-I[\Phi]} \, \delta\left(\Phi(\tau = 0^-, \mathbf{x} \in A) - \phi^A_-(\mathbf{x})\right) \delta\left(\Phi(\tau = 0^+, \mathbf{x} \in A) - \phi^A_+(\mathbf{x})\right) \,. \tag{46}$$

Note that everything that we have done so far in this subsection is valid for any choice of the entangling surface, $\partial A$. Let's now focus on the case where the entangling surface to be a codimension-2 infinite plane as in Eq. (30). The pictorial representation of the path integral in Eq. (46) is then shown in Fig. (4).

In the next subsection, we (will) compute the entanglement entropy of this reduced state.

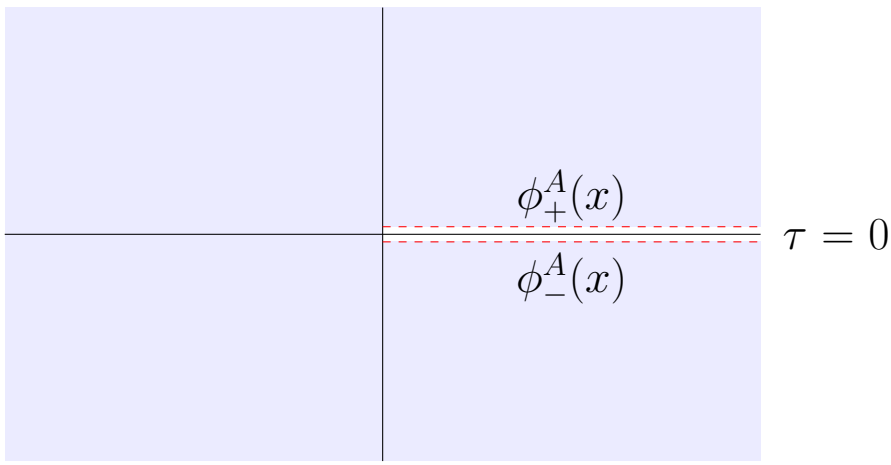


FIG. 4. Pictorial representation of Eq. (46) for the case when $A$: $x_\perp > 0$.

**Exercise 3** *Evaluate the path integral in Eq.* (38) *and show that the vacuum wavefunction for a free scalar field of mass $m$ is*

$$\Psi_0[\phi(\mathbf{x})] \sim \exp\Big(-\frac{1}{2}\int d^{d-1}\mathbf{x}\int d^{d-1}\mathbf{y}\,\phi(\mathbf{x})K(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})\Big)\,, \tag{47}$$

*where*

$$K(\mathbf{x}-\mathbf{y}) \;=\; \frac{1}{(2\pi)^{d-1}}\int d^{d-1}\mathbf{p}\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,\sqrt{\mathbf{p}^2+m^2}\,. \tag{48}$$

### B. Replica trick

The entanglement entropy of a reduced state is defined in Eq. (8). This involves a logarithm of a density matrix which is not easy to compute. Therefore, we do not use Eq. (8) to compute the entanglement entropy. Instead, we first compute the $n^{th}$ Rényi entropy, $S_A^{(n)}$, defined in Eq. (9), analytically continue it for non-integer $n$, and then take the limit $n \to 1$ as suggested in Eq. (12). To calculate $S_A^{(n)}$, we first need to find $\mathrm{tr}_A\,\rho_A^n$. This trace can also be written in terms of a path integral. To see this, note that the $\mathrm{tr}\,\rho_A^n$ can be written as

$$\mathrm{tr}\,\rho_A^n \;=\; \int\left(\prod_i^n D\phi_i^A\right)\;\langle\phi_1^A|\rho_A\,|\phi_2^A\rangle\langle\phi_2^A|\rho_A|\phi_3^A\rangle...\langle\phi_{n-1}^A|\rho_A|\phi_n^A\rangle\langle\phi_n^A|\rho_A|\phi_1^A\rangle\,. \tag{49}$$

This can be represented by taking n-copies of the path integral in Fig. (4), and gluing them together such that the $\tau = 0^+$ and $x_\perp > 0$ region on any copy is identified with the $\tau = 0^-$ and $x_\perp > 0$ region of the next copy. This means that $\mathrm{tr}\rho_A^n$ can be computed as the path

integral on an $n$-fold cover of our original Euclidean spacetime. That is,

$$\mathrm{tr}_A \rho_A^n = \frac{1}{Z_1^n} \int_{M_n} D\Phi \, e^{-I_n[\Phi]} \,, \tag{50}$$

$$\equiv \frac{Z_n}{Z_1^n} \,, \tag{51}$$

where we have denoted the replicated Euclidean space, that is $n$-fold cover of $R^d$, by $R_n^d$, and the action of our theory on $R_n^d$ by $I_n[\Phi]$. Now using Eq. (12), we get that the entanglement entropy of the reduced state of region $A$ is

$$S_A = -\frac{\partial}{\partial_n}\Big( \log Z_n - n \log Z_1 \Big)\Big|_{n=1} \,. \tag{52}$$

Before we do an explicit computation of $Z_n$, let's discuss the topology of the replicated space, $R_n^d$. Note that if we start from any point with coordinates $\tau = 0^+$ and $x_\perp > 0$ and rotate in the $\tau - x_\perp$ plane around the entangling surface by angle $2\pi$, we do not get to the same point. Instead, we reach the corresponding point in the next copy. This means we have to rotate by angle $2n\pi$ around the entangling surface to get to the same point in $R_n^d$. This implies that $R_n^d$ is a direct product of the form

$$R_n^d = C_n^2 \times R^{d-2} \,, \tag{53}$$

where $C_n^2$ is a cone of angle deficit of $2\pi(1-n)$. Using the polar coordinates to parameterize $C_n^2$, the metric on $R_n^d$ becomes

$$ds_n^2 = dr^2 + r^2 d\theta^2 + d\mathbf{x}_\parallel^2 \,, \qquad \theta \sim \theta + 2n\pi \,. \tag{54}$$

**Exercise 4** *Show that $C_n^2$ has a curvature singularity at the origin. In particular, show that the Ricci scalar of the metric*

$$ds_{C_n^2}^2 = dr^2 + r^2 d\theta^2 \,, \qquad \theta \sim \theta + 2n\pi \tag{55}$$

*is given by*

$$\mathcal{R}_{C_n^2} = -4\pi(n-1)\,\delta^2(\mathbf{x}) \,, \tag{56}$$

$$= -2\left(1 - \frac{1}{n}\right)\frac{1}{r}\delta(r) \,. \tag{57}$$

*Hint: Start with the above metric and introduce new coordinates: $\alpha = \theta/n$ and $r = \xi^n$. In these new coordinates, the metric looks conformally flat. Now use the conformal transformation of Ricci scalar to get the desired result.*

### C. Example: entanglement entropy of a free scalar field theory

In this subsection, we study an explicit example to show how to use the formalism of the last two subsections. Consider a free scalar field theory on a $d > 3$ dimensional flat spacetime

$$I[\Phi] = \frac{1}{2}\int_{R^d} d^dx\ (\partial\Phi)^2\,, \tag{58}$$

$$= \frac{1}{2}\int_{R^d} d^dx\ \Phi\left(-\partial^2\right)\Phi\,. \tag{59}$$

Now we define the action of this theory on a replicated space, $R^d_n$, by minimal coupling. That is,

$$I_n[\Phi] = \frac{1}{2}\int_{R^d_n} d^dx\,\sqrt{g}\ \Phi\left(-\nabla^2\right)\Phi\,, \tag{60}$$

where $\nabla^2$ is a Laplacian on $R^d_n$. The partition function of this theory is given by

$$\log Z_n = -\frac{1}{2}\log\det\left(-\nabla^2\right)\,, \tag{61}$$

$$= -\frac{1}{2}\ \mathrm{tr}\ \log\left(-\nabla^2\right)\,. \tag{62}$$

$$= \frac{1}{2}\int_{\delta^2}^{\infty}\frac{ds}{s}\ \mathrm{tr}\, e^{-s\left(-\nabla^2\right)}\,, \tag{63}$$

where we have introduced a small length scale $\delta$ to regulate the divergences in this integral.

To proceed, we first need to *diagonalize* the Laplacian operator on $R^d_n$. We already know that that the Laplacian on $R^{d-2}$ is diagonalized by plane waves. Now we only have to diagonalize the Laplacian on $C^2_n$. Let's assume that

$$F_{\lambda,q}(r,\theta) = \frac{1}{\sqrt{2\pi n}}\ e^{iq\theta/n}\, f_\lambda(r)\,, \qquad \text{for } q\in Z\,, \tag{64}$$

is an eigenfunction of $-\nabla^2_{C^2_n}$ with eigenvalue $\lambda^2$. That is

$$-\nabla^2_{C^2_n} F_{\lambda,q}(r,\theta) = \lambda^2 F_{\lambda,q}(r,\theta)\,, \tag{65}$$

or equivalently

$$r^2 f''_\lambda(r) + r f'_\lambda(r) + \left(\lambda^2 r^2 - q^2/n^2\right) f_\lambda(r) = 0\,, \tag{66}$$

where prime denotes the derivative with respect to $r$. This is a Bessel equation. Demanding that the solutions of this equation are regular at $r = 0$, we get

$$f_\lambda(r) = \sqrt{\lambda}\, J_{|q/n|}(\lambda r)\,. \tag{67}$$

Therefore,

$$\chi_{\lambda,q,\mathbf{p}}(r,\theta,\mathbf{y}) = \frac{1}{(2\pi)^{(d-2)/2}}\sqrt{\frac{\lambda}{2\pi n}}\, e^{i\mathbf{p}\cdot\mathbf{y}}\, e^{iq\theta/n}\, J_{|q/n|}(\lambda r) \tag{68}$$

is an eigenfunction of the Laplacian on $R^d_n$ with eigenvalue $-(\lambda^2+\mathbf{p}^2)$. The overall normalization ensures that these eigenfunctions are orthonormal. That is, [6]

$$\int d^{d-2}\mathbf{y}\int_0^\infty dr\, r\int_0^{2\pi n} d\theta\; \chi^*_{\lambda,q,\mathbf{p}}(r,\theta,\mathbf{y})\,\chi_{\lambda',q',\mathbf{p}'}(r,\theta,\mathbf{y}) \;=\; \delta_{qq'}\,\delta(\lambda-\lambda')\,\delta(\mathbf{p}-\mathbf{p}')\,. \tag{69}$$

Now we write the trace in Eq. (63) as

$$\begin{aligned}
\mathrm{tr}\, e^{-s\left(-\nabla^2\right)} &= \sum_q \int d^{d-2}\mathbf{p}\int_0^\infty d\lambda\;\int d^{d-2}\mathbf{y}\int_0^\infty dr\, r\int_0^{2\pi n} d\theta && (70)\\
&\qquad\times \chi^*_{\lambda,q,\mathbf{p}}(r,\theta,\mathbf{y})\, e^{-s\left(-\nabla^2\right)}\,\chi_{\lambda,q,\mathbf{p}}(r,\theta,\mathbf{y})\,, && (71)\\
&= \sum_q \int_0^\infty dr\, r\int_0^\infty d\lambda\,\lambda\, e^{-s\lambda^2}\,\left(J_{|q/n|}(\lambda r)\right)^2 && (72)\\
&\qquad\times \frac{1}{(2\pi)^{(d-2)}}\int d^{d-2}\mathbf{y}\int d^{d-2}\mathbf{p}\, e^{-s\mathbf{p}^2}\,, && (73)\\
&= \frac{\mathcal{A}}{(4\pi s)^{(d-2)}}\sum_q \int_0^\infty dr\, r\int_0^\infty d\lambda\,\lambda\, e^{-s\lambda^2}\,\left(J_{|q/n|}(\lambda r)\right)^2\,, && (74)
\end{aligned}$$

where we have defined

$$\mathcal{A} \equiv \int d^{d-2}\mathbf{y}\,, \tag{75}$$

as the area of the entangling surface. Now using the identity for $Re(\alpha) > -1$

$$\int_0^\infty d\lambda\,\lambda\, e^{-s\lambda^2}\,\left(J_\alpha(\lambda r)\right)^2 = \frac{1}{2s}e^{-\frac{r^2}{2s}}\, I_\alpha\left(\frac{r^2}{2s}\right)\,, \tag{76}$$

we simplify Eq. (74) to get

$$\begin{aligned}
\mathrm{tr}\, e^{-s\left(-\nabla^2\right)} &= \frac{2\pi\mathcal{A}}{(4\pi s)^{d/2}}\sum_q\int_0^\infty dr\, r\, e^{-\frac{r^2}{2s}}\, I_{|q/n|}\left(\frac{r^2}{2s}\right)\,, && (77)\\
&= \frac{\mathcal{A}}{(4\pi s)^{(d-2)/2}}\sum_q\int_0^\infty dz\, z\, e^{-z^2}\, I_{|q/n|}\left(z^2\right)\,. && (78)
\end{aligned}$$

This integral diverges for large $z$. To regulate this IR divergence, we note that for sufficiently large $w$,

$$\int_0^\infty dz\, z^{-w}\, e^{-z^2}\, I_{|q/n|}\left(z^2\right) = 2^{(w-3)/2}\,\frac{\Gamma(w/2)}{\sqrt{\pi}}\,\frac{\Gamma((1-w)/2+|q/n|)}{\Gamma((1+w)/2+|q/n|)}\,. \tag{79}$$

Taking the limit $w\to -1$ yields

$$\int_0^\infty dz\, z\, e^{-z^2}\, I_{|q/n|}\left(z^2\right) = -\,\frac{1}{2}\,|q/n|\,. \tag{80}$$

With this, we write Eq. (78) as

$$\operatorname{tr} e^{-s\left(-\nabla^2\right)} = -\frac{\mathcal{A}}{(4\pi s)^{(d-2)/2}}\frac{1}{2}\sum_q |q/n|\,, \tag{81}$$

$$= -\frac{\mathcal{A}}{(4\pi s)^{(d-2)/2}}\frac{1}{n}\zeta(-1)\,, \tag{82}$$

$$= \frac{\mathcal{A}}{(4\pi s)^{(d-2)/2}}\frac{1}{12n}\,. \tag{83}$$

Inserting this result in Eq. (63) yields

$$\log Z_n = \frac{\mathcal{A}}{\epsilon^{d-2}} \quad \frac{1}{12n(d-2)(4\pi)^{(d-2)/2}}\,. \tag{84}$$

Now using Eq. (52), we get

$$S_A = \frac{\mathcal{A}}{\epsilon^{d-2}} \quad \frac{1}{6(d-2)(4\pi)^{(d-2)/2}}\,. \tag{85}$$

**Exercise 5** *Repeat the analysis for a free scalar field theory of mass $m$.*

### D. Area law of the entanglement entropy

As we saw in the last subsection, the entanglement entropy of a half-space in a vacuum state of a scalar field theory is a UV divergent quantity. Moreover, we found that the divergent part of the entanglement entropy is proportional to the area of the entangling surface. The proportionality of the UV divergences to the area of the entangling surface, $\partial A$, rather than the volume of the region $A$ suggests that these divergences are due to the infinite short distance entanglement between nearby modes residing on either side of the entangling surface. In this sense, the UV divergences in the entanglement entropy are *local* to the entangling surface.

The local nature of the UV divergences in the entanglement entropy has interesting consequences. It implies that the leading UV divergence should scale as the area of the entangling surface for any choice of the entangling surface. Furthermore, since any curved spacetime locally looks like a flat spacetime, we deduce that the leading divergence in the entanglement entropy for an arbitrary entangling surface in a ($d \geq 3$)-dimensional curved spacetime is of the form

$$S \sim \frac{\mathcal{A}}{\delta^{d-2}} + \ldots\,, \tag{86}$$

where $\mathcal{A}$ is the area of the entangling surface, and $\delta$ is a short distance UV cutoff. The proportionality constant depends on the details of the field theory, entangling surface, and the choice of regularization scheme.

Even though entanglement entropy of any spatial region is a divergent quantity, the mutual information of two non-adjacent regions, defined in Eq. (24), is a UV finite quantity. The cancellation of the UV divergences in the mutual information is another important result that follows from the locality of UV divergences.

## III. ENTANGLEMENT ENTROPY IN $(1+1)$-DIMENSIONAL CFT

In this section, we focus on the $(1+1)$-dimensional conformal field theories. We show in this section that the conformal invariance implies the entanglement entropy of some interval of size $\ell$ is a universal result, and it only depends on the central charge. We further show that this entanglement entropy is related to the conformal anomaly. Though we only show this for $(1+1)$-dimensional CFTs, this connection between the entanglement entropy and conformal anomalies is also true in higher dimensions.

We are interested in a CFT in $R^2$. Let's take subregion $A$ to be a single interval, $x_1 \leq x \leq x_2$, of length $\ell \equiv |x_2 - x_1|$. The entanglement entropy of this region, $S_A$, can be computed using the Replica trick. This involves defining the theory on a replicated space with conical singularities on the end points of the region $A$, that is, $x = x_1$ and $x = x_2$. From Eq. (52), we get

$$S_A(\ell) = \log Z_1 - \lim_{n\to 1} \frac{\partial}{\partial n} \log Z_n \,. \tag{87}$$

To find the dependence of $S_A$ on the size of the region $A$, we rescale $\ell$ by an infinitesimal amount,

$$\ell \to e^{\omega}\, \ell \approx (1+\omega)\, \ell \,. \tag{88}$$

This is equivalent to a constant infinitesimal Weyl scaling

$$g_{ab} \to e^{2\omega}\, g_{ab} \approx (1+2\omega) g_{ab} \,. \tag{89}$$

Since there is no other length scale in the theory, we deduce that

$$\ell \frac{d}{d\ell} S_A(\ell) = 2 \int d^2x \; g_{ab} \; \frac{\delta}{\delta g_{ab}} S_A(\ell) \,. \tag{90}$$

Now note that

$$\frac{\delta}{\delta g_{ab}} \log Z_n = \frac{1}{Z_n} \frac{\delta}{\delta g_{ab}} Z_n \,, \tag{91}$$

$$= \frac{1}{Z_n} \frac{\delta}{\delta g_{ab}} \int_{R_n^2} D\Phi \, e^{-I_n[\Phi;g]} \,, \tag{92}$$

$$= - \frac{1}{Z_n} \int_{R_n^2} D\Phi \, e^{-I_n[\Phi;g]} \, \frac{\delta}{\delta g_{ab}} I_n[\Phi; g] \,, \tag{93}$$

$$= - \frac{1}{Z_n} \int_{R_n^2} D\Phi \, e^{-I_n[\Phi;g]} \, \frac{\sqrt{g}}{2} T^{ab} \,, \tag{94}$$

$$= - \frac{\sqrt{g}}{2} \left\langle T^{ab} \right\rangle_{R_n^2} . \tag{95}$$

We combine this result with Eq. (87) and insert it into Eq. (90) to get

$$\ell \frac{d}{d\ell} S_A(\ell) \, = \, - \int_{R^2} d^2x \, \sqrt{g} \left\langle T^a{}_a \right\rangle_{R^2} + \lim_{n \to 1} \frac{\partial}{\partial n} \int_{R_n^2} d^2x \, \sqrt{g} \left\langle T^a{}_a \right\rangle_{R_n^2} . \tag{96}$$

Classically, the conformal invariance implies that the stress-tensor must be trace-less. However, there is an anomaly in quantum theory in curved spacetime. The conformal anomaly in $(1+1)$-dimensions states that

$$\left\langle T^a{}_a \right\rangle \, = \, - \frac{c}{24\pi} \mathcal{R} \,, \tag{97}$$

where $\mathcal{R}$ is the Ricci scalar of the background spacetime. Since $R^2$ is Ricci flat, the first term in Eq. (96) vanishes. The second term does not vanish due to the conical singularities at the endpoints of region $A$ in $R_n^2$ (see exercise (4). The Ricci scalar, in this case, is given by (see Eq. (56))

$$\mathcal{R}_n = \, - 4\pi(n-1) \Big( \delta(\tau)\delta(x - x_1) + \delta(\tau)\delta(x - x_2) \Big) \,, \tag{98}$$

With this result, Eq. (96) becomes

$$\ell \frac{d}{d\ell} S_A(\ell) \, = \, \frac{c}{3} \,, \tag{99}$$

which gives us

$$S_A(\ell) = \, \frac{c}{3} \log \frac{\ell}{\delta} \, + \, c_1 \,, \tag{100}$$

where $\delta$ is a UV cutoff, and $c_1$ is a constant that depends on the CFT and on the regularization scheme. Note that unlike $c_1$, the coefficient of the logarithm term is fixed by conformal invariance and hence, it is same for *all* CFTs.

## IV. PERTURBATIVE METHODS

The computation of the entanglement entropy for a general field theory is a difficult exercise. In this section, we present a perturbative method of [7–9] to calculate the entanglement entropy. Consider a field theory with action $I_0$ and perturb it by a relevant operator, $\mathcal{O}$. Then the action of the perturbed theory is given by

$$I_\lambda = I_0 + \lambda \mathcal{O} \tag{101}$$

where

$$\mathcal{O} = \int d^d x \, \mathcal{O}(x) \, , \tag{102}$$

and $\lambda$ is the coupling constant. Our goal in this section is to write the entanglement entropy for *any* subregion as a power series in $\lambda$. That is,

$$S_A(\lambda) = S_A(0) + \lambda \, \frac{dS_A(\lambda)}{d\lambda}\Big|_{\lambda=0} + \frac{\lambda^2}{2} \, \frac{d^2 S_A(\lambda)}{d\lambda^2}\Big|_{\lambda=0} + \dots \, . \tag{103}$$

To do this, we use the modular Hamiltonian formula, Eq. (20), for the entanglement entropy. Using Eq. (20), the entanglement entropy of any subregion $A$ in a vacuum state of the perturbed theory can be written as

$$S_A(\lambda) = \langle K_A(\lambda) \rangle_\lambda \, , \tag{104}$$

where $\langle ... \rangle_\lambda = {}_\lambda\langle 0 | ... | 0 \rangle_\lambda$, and $K_A(\lambda)$ is the modular Hamiltonian for the region $A$. We can write this vacuum expectation value as

$$S_A(\lambda) \, = \, \frac{1}{Z_\lambda} \int D\Phi \, e^{-I_\lambda} \, K_A(\lambda) \, . \tag{105}$$

We take the derivative of this equation to get

$$\frac{dS_A(\lambda)}{d\lambda} = \frac{1}{Z_\lambda} \int D\Phi \, e^{-I_\lambda} \Big[ - \frac{dI_\lambda}{d\lambda} \, K_A(\lambda) \, + \frac{dK_A(\lambda)}{d\lambda} - \frac{d \log Z_\lambda}{d\lambda} \, K_A(\lambda) \Big] \, , \tag{106}$$

Now using

$$\frac{dI_\lambda}{d\lambda} = \mathcal{O} \, , \tag{107}$$

and

$$\frac{d \log Z_\lambda}{d\lambda} = - \langle \mathcal{O} \rangle_\lambda \, , \tag{108}$$

we get

$$\frac{dS_A(\lambda)}{d\lambda} = \frac{1}{Z_\lambda} \int D\Phi\, e^{-I_\lambda} \Big[ -\mathcal{O}\, K_A(\lambda) + \frac{dK_A(\lambda)}{d\lambda} + \langle \mathcal{O} \rangle_\lambda\, K_A(\lambda) \Big] , \tag{109}$$

$$= -\langle \mathcal{O} K_A(\lambda) \rangle_\lambda + \langle \mathcal{O} \rangle_\lambda \langle K_A(\lambda) \rangle_\lambda + \left\langle \frac{dK_A(\lambda)}{d\lambda} \right\rangle_\lambda , \tag{110}$$

$$= -\langle \mathcal{O} K_A(\lambda) \rangle_{c,\lambda} + \left\langle \frac{dK_A(\lambda)}{d\lambda} \right\rangle_\lambda . \tag{111}$$

Note that the second term in Eq. (111) identically vanishes. This follows from the normalization of the density matrix, $\mathrm{tr}_A \rho_A = 1$:

$$\left\langle \frac{dK_A(\lambda)}{d\lambda} \right\rangle_\lambda = \mathrm{tr}_{A\cup\bar{A}} \Big[ \rho(\lambda)\, \frac{dK_A(\lambda)}{d\lambda} \Big] , \tag{112}$$

$$= \mathrm{tr}_A \Big[ \rho_A(\lambda)\, \frac{dK_A(\lambda)}{d\lambda} \Big] , \tag{113}$$

$$= \mathrm{tr}_A \Big[ e^{-K_A(\lambda)}\, \frac{dK_A(\lambda)}{d\lambda} \Big] , \tag{114}$$

$$= -\frac{d}{d\lambda}\, \mathrm{tr}_A e^{-K_A(\lambda)} , \tag{115}$$

$$= 0 . \tag{116}$$

Therefore, we get

$$\frac{dS_A(\lambda)}{d\lambda} = -\langle \mathcal{O} K_A(\lambda) \rangle_{c,\lambda} . \tag{117}$$

Now taking the derivative of this equation and repeating the above analysis yield

$$\frac{d^2 S_A(\lambda)}{d\lambda^2} = \langle \mathcal{O}\mathcal{O} K_A(\lambda) \rangle_{c,\lambda} - \left\langle \mathcal{O}\, \frac{dK_A(\lambda)}{d\lambda} \right\rangle_\lambda . \tag{118}$$

Note that these results are applicable for any spatial subregion $A$. However, the modular Hamiltonian is not known for a general region. Hence, these formulas are not practical to compute the entanglement entropy for an arbitrary region. Let's now focus on the case where $A$ is a half-space as in Eq. (31). In this case, the modular Hamiltonian is related to the boost operator (see Eq. (32)). More precisely,

$$K_A(\lambda) = 2\pi H_A(\lambda) + \log Z_A(\lambda) , \tag{119}$$

where $H_A$ is a boost operator and is given in Eq. (33):

$$H_A(\lambda) = \int d^{d-2}x_\perp \int_0^\infty dx_1\, x_1\, T_{00}^{(\lambda)}(t=0, x_1, \mathbf{x}_\perp) . \tag{120}$$

Note that $\log Z_A(\lambda)$ is just a 'number' and it drops out from the connected correlation functions in Eq. (117) and Eq. (118).

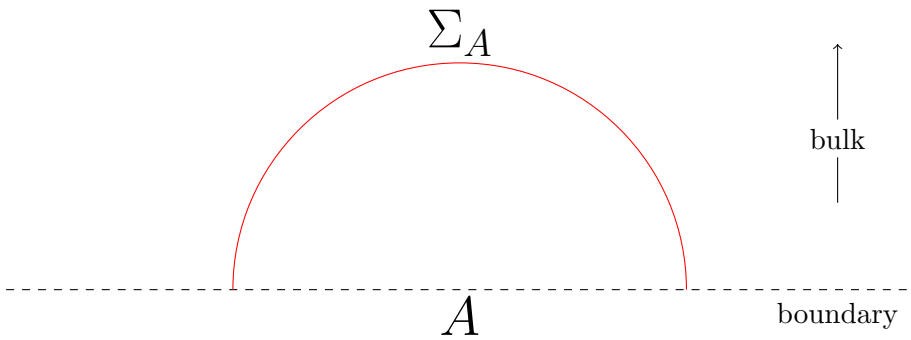


FIG. 5. The entanglement entropy of region $A$ is given by the area of the codimension-2 minimal area bulk surface, $\Sigma_A$ anchored at the endpoints of $A$.

**Exercise 6** *Use Eq.* (117) *to show that the entanglement entropy of a free scalar field of mass $m$ has a logarithmic divergence in $d = 4$. Find the coefficient of this term and compare it with your result of exercise (5).*

**Exercise 7** *Use Eq.* (118) *to show that the entanglement entropy of a free scalar field of mass $m$ has a logarithmic divergence in $d = 4$. Find the coefficient of this term.*

## V. HOLOGRAPHIC METHODS

The calculation of the entanglement entropy for a general entangling surface is not an easy task. This involves first finding the reduced density matrix and then diagonalizing it. For field theories with a classical holographic dual, the AdS-CFT correspondence provides an alternative prescription. The holographic formula for the entanglement entropy of any spatial region $A$, is [10]

$$S_A = \frac{A(\Sigma_A)}{4G}\,, \tag{121}$$

where $A(\Sigma_A)$ is the area of a codimension-2 spacelike stationary area surface, $\Sigma_A$, in the bulk subject to the conditions that it is anchored on the boundary at the entangling surface, $\partial A$, and that it is homologous to the boundary region $A$ (See Fig. (5)). If there are several such surfaces, we choose the one with the minimum area. The holographic formula, Eq. (121), was first derived in [11] for time-independent states and then in [12] for general states.

In addition to providing an analytical tool to compute the entanglement entropy for a general entangling surface, this holographic formula connects a quantum information quantity

to a geometric quantity. This connection has led to many new insights about the emergence of classical spacetime [13, 14]. Moreover, it has been shown that the holographic entanglement entropy satisfies several inequalities that are not valid in general [15, 16]. Therefore, these inequalities can be used to determine what CFTs have a classical gravitational bulk dual.

In the special case when the QFT state is time-independent and the bulk geometry is stationary, the stationary area surface $\Sigma_A$ becomes a *minimal* area surface on a bulk Cauchy slice [17]. In this section, we will only focus on time-independent states.

### A. Entanglement entropy of a single interval in $(1+1)$ dimensions

Suppose we are in a vacuum state of a $(1+1)$-dimensional CFT in $R^2$. Let's take region $A$ to be an interval of size $\ell$,

$$A: \, -\ell/2 \leq x \leq \ell/2 \,, \quad \text{and} \quad t = 0 \,. \tag{122}$$

To compute the entanglement entropy of region $A$, we need to find a spacelike codimension-2 surface of minimal area in $\mathrm{AdS}_3$ spacetime. The metric of $\mathrm{AdS}_3$ (in Poincaré coordinates) is given by [1]

$$ds^2 = \frac{1}{z^2}\left(dz^2 - dt^2 + dx^2\right) \,, \tag{123}$$

where $z = 0$ is the conformal boundary. Note that a codimension-2 surface of the minimal area in $\mathrm{AdS}_3$ is simply a geodesic. Furthermore, due to time translation invariance, the surface $\Sigma_A$ must lie on a $t = 0$ slice in the bulk.

Now consider a general spacelike surface, $\gamma_A$, that lies in the $t = 0$ slice of $\mathrm{AdS}_3$ and is anchored on the endpoints of $A$. This surface can be described using the parametric equation

$$\gamma_A: \, z = z(x) \,, \quad \text{and} \quad t = t(x) = 0 \,, \tag{124}$$

with the boundary conditions

$$z(\pm\ell/2) = 0 \,. \tag{125}$$

The induced metric on this surface is given by

$$ds^2_{\gamma_A} = \frac{1}{z^2}\left(1 + (z'(x))^2\right) dx^2 \,, \tag{126}$$

[1] We set $\mathrm{L}_{AdS} = 1$ .

where prime denotes derivative with respect to $z$. The area of this surface is given by

$$\mathcal{A}(\gamma_A) = \int_{-\ell/2}^{\ell/2} dx \, \frac{1}{z(x)} \sqrt{1 + (z'(x))^2} \, . \tag{127}$$

We need to find $z(x)$ that minimizes this area functional. This is a standard *calculus of variation* problem. Since the 'Lagrangian' is independent of $x$, we deduce that the 'Hamiltonian' must be a constant. This gives us the following equation

$$z^2 \left(1 + (z'(x))^2\right) = z_*^2 \, , \tag{128}$$

where $z_* = z(0)$. The solution of this equation is

$$z(x) = \sqrt{z_*^2 - x^2} \, . \tag{129}$$

Now imposing the boundary conditions in Eq. (125), we find that the minimal area surface anchored at the endpoints of the region $A$ is

$$\Sigma_A : \, z = \sqrt{(\ell/2)^2 - x^2} \, , \tag{130}$$

and its area is given by

$$\mathcal{A}(\Sigma_A) = \int_{-\ell/2}^{\ell/2} dx \, \frac{\ell/2}{(\ell/2)^2 - x^2} \, , \tag{131}$$

$$= \int_{0}^{\ell/2} dx \, \frac{\ell}{(\ell/2)^2 - x^2} \, , \tag{132}$$

$$= \int_{0}^{\ell/2-\delta} dx \, \frac{1}{\ell/2 - x} + \int_{0}^{\ell/2} dx \, \frac{1}{\ell/2 + x} \, . \tag{133}$$

Note that the first integral in the last expression diverges. To regulate this integral, we introduce a cut-off surface near the boundary of $\mathrm{AdS}_3$ at $z = \delta$. This means we only integrate till $x = \frac{\ell}{2} - \frac{\delta^2}{\ell}$. With this, the above integral becomes

$$\mathcal{A}(\Sigma_A) = \int_{0}^{\ell/2-\delta^2/\ell} dx \, \frac{1}{\ell/2 - x} + \int_{0}^{\ell/2} dx \, \frac{1}{\ell/2 + x} \, , \tag{134}$$

$$= 2 \, \log \frac{\ell}{\delta} \, , \tag{135}$$

The entanglement entropy of region $A$ according to the holographic formula, Eq. (121), is the given by

$$S_A = \frac{1}{4G} \, \log \frac{\ell}{\delta} \, , \tag{136}$$

$$= \frac{c}{3} \, \log \frac{\ell}{\delta} \, , \tag{137}$$

where we have used Brown-Henneaux [18] result

$$c = \frac{3}{2G} \,. \tag{138}$$

This matches the field theory prediction in Eq. (100).

Before we proceed, we make two important observations:

1. The spacelike codimension-2 corresponding to the spatial region $\bar{A}$ is same as that corresponding to the region $A$. This implies that

$$S_A = S_{\bar{A}} \,, \tag{139}$$

which is consistent with Eq. (21) for pure states.

2. In Sec. (II D), we saw that the UV divergences in the entanglement entropy are due to the infinite entanglement between modes living near the entangling surface. In the holographic setting, the UV divergences arise because the area of a surface diverges near the boundary ($z = 0$) due to a conformal factor of $1/z^2$ in the metric. Since the stationary area surface, $\Sigma_A$, is anchored on the boundary at $\partial A$, the divergences in the entanglement entropy must be proportional to the area of $\partial A$.

### B. Holographic entanglement inequalities

Here, we show that the holographic entanglement entropy satisfies subadditivity, Eq. (23), and strong subadditivity [19, 20], Eq. (25), and monogamy of mutual information [15]. For simplicity, we will only focus on time-independent states of a $(1 + 1)$-dimensional CFTs.

#### 1. *Holographic proof of subadditivity*

Now let's consider two disjointed subregions, $A_1 : x_1 \leq x \leq x_2$ and $A_2 : x_3 \leq x \leq x_4$. The entanglement entropy of the union of these subregions, $S_{A_1 \cup A_2}$, is given by the area of the minimal area surface anchored on the endpoints on $A_1$ and $A_2$. However, there are two possible minimal area surfaces as shown in Fig. (6). First, there is the union of the bulk minimal surfaces associated to the two intervals $[x_1, x_2]$ and $[x_3, x_4]$ (shown as red surfaces in Fig. (6)). Second, there is the union of the bulk minimal surfaces associated to

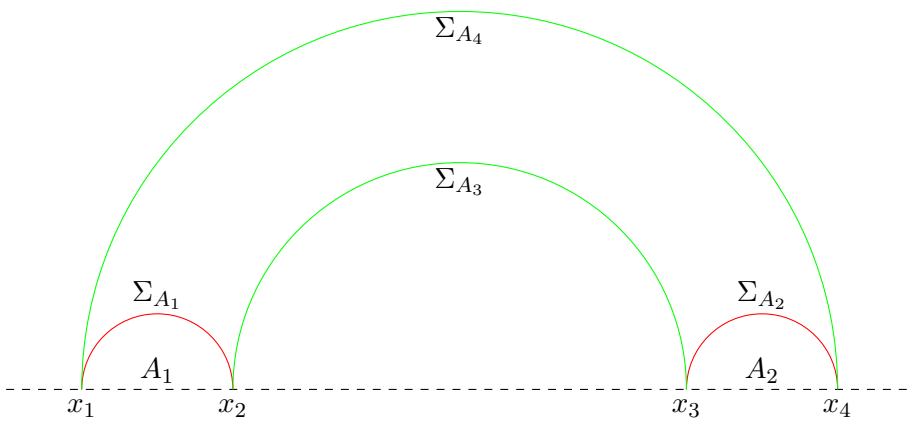


FIG. 6. There are two candidates for minimal area surfaces corresponding to the boundary region $A_1 \cup A_2$.

the two intervals $[x_1, x_4]$ and $[x_2, x_3]$ (shown as green surfaces in Fig. (6)). The holographic formula dictates that we compute the total area in both cases and take the minimum value. Therefore, we get

$$S_{A_1 \cup A_2} = \frac{1}{4G} \times \min\Big\{\mathcal{A}(\Sigma_{A_1}) + \mathcal{A}(\Sigma_{A_2})\,,\, \mathcal{A}(\Sigma_{A_3}) + \mathcal{A}(\Sigma_{A_4})\Big\}\,. \tag{140}$$

In general, finding a minimum of two divergent quantities might not be a well-defined problem. Luckily in our case, the divergences in the area of the red surfaces are same as the divergences in the area of the green surfaces. This follows from the facts that the divergences in the area of these surfaces are determined by the geometry near the boundary, and the red and green surfaces have the same boundary conditions. Now the above equation implies

$$S_{A_1 \cup A_2} \leq \frac{\mathcal{A}(\Sigma_{A_1})}{4G} + \frac{\mathcal{A}(\Sigma_{A_2})}{4G}\,, \tag{141}$$

which is equivalent to

$$S_{A_1 \cup A_2} \leq S_{A_1} + S_{A_2}\,. \tag{142}$$

This is the subadditivity property of entanglement entropy, Eq. (23).

### 2. *Holographic proof of strong subadditivity*

Now let's consider two overlapping subregions, $A_1 : x_1 \leq x \leq x_2$ and $A_2 : x_3 \leq x \leq x_4$, as shown in Fig. (7). The minimal area surfaces corresponding to the boundary region $A_1$ and $A_2$ are shown as the red surfaces in Fig. (7), whereas the green surfaces denote the

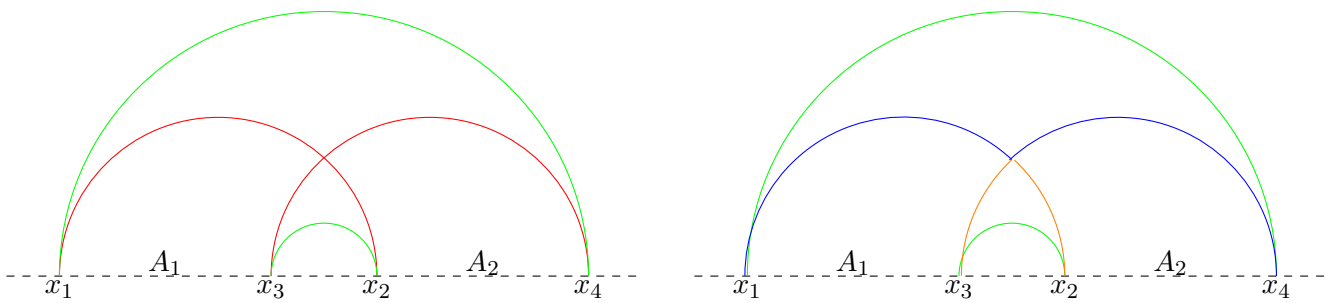


FIG. 7. Pictorial proof of the strong subadditivity.

minimal area surfaces corresponding to the boundary region $A_1 \cup A_2$ and $A_1 \cap A_2$. The sum of the entanglement entropy of boundary regions $A_1$ and $A_2$ is given by the sum of the red surfaces. That is,

$$S_{A_1} + S_{A_2} = \frac{1}{4G}\,\mathcal{A}(\text{red surfaces})\,. \tag{143}$$

Note that the sum of the areas of the two red surfaces is same as the sum of the area of the blue and orange surfaces in Fig. (7). Therefore

$$\mathcal{A}(\text{red surfaces}) = \mathcal{A}(\text{blue surface}) + \mathcal{A}(\text{orange surface})\,. \tag{144}$$

Now observe that the blue surface is a bulk surface anchored on the endpoints on the region $A_1 \cup A_2$, and its area must be greater than the area of the minimal area surface corresponding to the region $A_1 \cup A_2$. Similarly, the area of the orange surface must be greater than the area of the minimal area surface corresponding to the region $A_1 \cap A_2$. This implies that the sum of the area of the blue and the orange surfaces is greater than the sum of the green surfaces. That is

$$\mathcal{A}(\text{blue surface}) + \mathcal{A}(\text{orange surface}) \;\geq\; \mathcal{A}(\text{green surfaces})\,. \tag{145}$$

The area of the green surfaces is proportional to the entanglement entropy of boundary regions $A_1 \cup A_2$ and $A_1 \cap A_2$,

$$S_{A_1 \cup A_2} + S_{A_1 \cap A_2} = \frac{1}{4G}\,\mathcal{A}(\text{green surfaces})\,. \tag{146}$$

Now we combine Eqs. (143)-(146) to get

$$S_{A_1 \cup A_2} + S_{A_1 \cap A_2} \leq S_{A_1} + S_{A_2}\,. \tag{147}$$

This is the strong subadditivity, Eq. (25).

### 3. Monogamy of mutual information

Now we consider three intervals $A$, $B$, and $C$. Let's, for simplicity, take these regions to be adjacent. That is, $A : x_1 \leq x \leq x_2$, $B : x_2 \leq x \leq x_3$, and $C : x_3 \leq x \leq x_4$, as shown in Fig. (8). The minimal area surfaces corresponding to the region $A \cup B$ and $B \cup C$ are shown as red surfaces. Therefore,

$$S_{A\cup B} + S_{B\cup C} = \frac{1}{4G}\,\mathcal{A}(\text{red surfaces})\,. \tag{148}$$

For the region $A \cup C$, there are two candidates for the minimal area surfaces. These are either green surfaces or the black surfaces. Therefore

$$S_{A\cup C} = \frac{1}{4G} \times \min\Big\{\mathcal{A}(\text{green surfaces})\,,\,\mathcal{A}(\text{black surfaces})\Big\}\,. \tag{149}$$

We consider these two cases separately. Let the area of the black surfaces is lesser than the area of the green surfaces. Then we get

$$S_{A\cup C} = \frac{1}{4G}\,\mathcal{A}(\text{black surfaces})\,, \tag{150}$$

$$= S_A + S_C\,. \tag{151}$$

With this result, we get

$$S_{A\cup B} + S_{B\cup C} + S_{A\cup C} = S_{A\cup B} + S_{B\cup C} + S_A + S_C\,, \tag{152}$$

$$\geq S_A + S_B + S_C + S_{ABC}\,, \tag{153}$$

where we have used the strong subadditivity, Eq. (25), in the last step.

Now let's assume that the area of the green surfaces is lesser than the area of the black surfaces. With this assumption, Eq. (149) becomes

$$S_{A\cup C} = \frac{1}{4G}\,\mathcal{A}(\text{green surfaces})\,, \tag{154}$$

$$= S_B + S_{ABC}\,. \tag{155}$$

Adding this result to Eq. (148), we get

$$S_{A\cup B} + S_{B\cup C} + S_{A\cup C} = \frac{1}{4G}\,\mathcal{A}(\text{red surfaces}) + S_B + S_{ABC}\,. \tag{156}$$

Note that the area of the red surfaces is same as the sum of the blue and the orange surface. Therefore,

$$S_{A\cup B} + S_{B\cup C} + S_{A\cup C} = \frac{1}{4G}\,\mathcal{A}(\text{blue surface}) + \frac{1}{4G}\,\mathcal{A}(\text{orange surface}) + S_B + S_{ABC}\,. \tag{157}$$

Now note that the blue surface is a bulk surface anchored at the endpoints of region $A$. Therefore, its area must be greater than the area of a black surface anchored at the endpoints of $A$. Similarly, the area of the orange surface must be greater than that of the black surface anchored at the endpoints of $C$. Therefore, we get

$$S_{A\cup B} + S_{B\cup C} + S_{A\cup C} \geq \frac{1}{4G}\,\mathcal{A}(\text{black surfaces}) + S_B + S_{ABC}\,, \tag{158}$$

$$= S_A + S_B + S_C + S_{ABC}\,. \tag{159}$$

Hence, we have derived an inequality

$$S_{A\cup B} + S_{B\cup C} + S_{A\cup C} \geq\ S_A + S_B + S_C + S_{ABC}\,, \tag{160}$$

which can be written as

$$I(A : B\cup C) \geq\ I(A : B) + I(A : C)\,. \tag{161}$$

This inequality is called *monogamy of mutual information.*

An interesting point about this holographic inequality is that it is not valid for a general quantum system [**?** ]. Therefore, this inequality provides a necessary condition for a quantum system to have a classical holographic dual.

**Exercise 8** *Show that the monogamy of mutual information still holds even when the regions $A$, $B$, and $C$ are not adjacent.*

**Exercise 9** *Suppose a system of three qubits is in a mixed state*

$$\rho = \frac{1}{2}|000\rangle\langle 000| + \frac{1}{2}|111\rangle\langle 111|\,. \tag{162}$$

*Verify the that monogamy of mutual information is not satisfied in this state.*

### C. Bekenstein-Hawking entropy as thermal entropy

In this section, we assume that the boundary CFT is in a thermal state. The thermal state in the boundary theory is dual to a black hole in the bulk. The thermal entropy of the boundary state is defined as the von Neumann entropy of the thermal density matrix. According to the holographic formula, Eq. (121), this entropy is related to the area of a

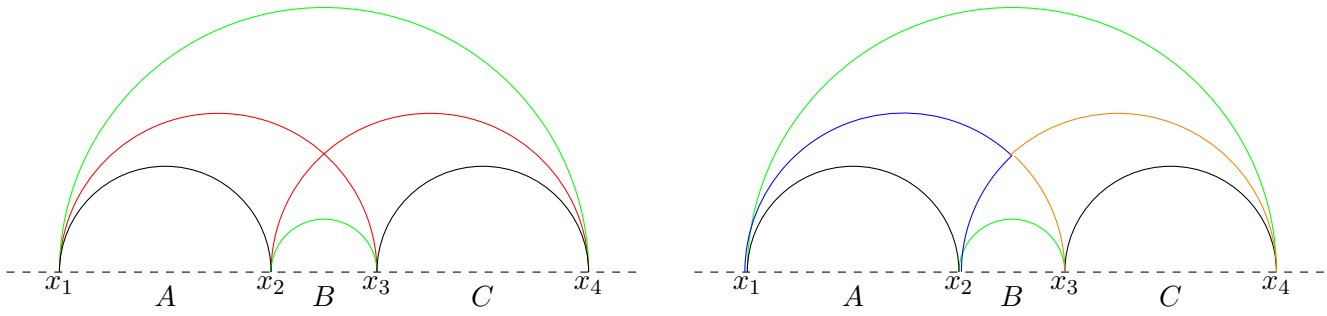


FIG. 8. Pictorial proof of the monogamy of mutual information.

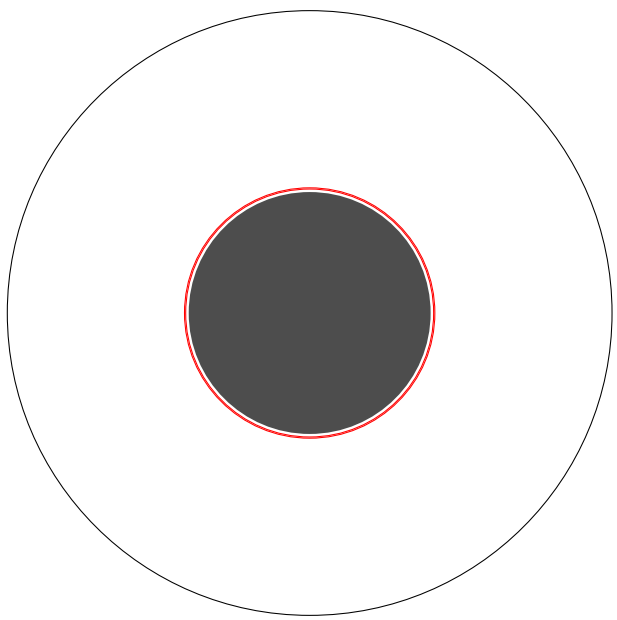

FIG. 9. The minimal area surface corresponding to the entire boundary region in the event horizon of a black hole in the bulk.

surface homologous to the entire boundary. Note that the codimension-2 spacelike minimal area surface homologous to the entire boundary is the event horizon. Therefore, Eq. (121) reduces to

$$S_{\text{thermal}} = \frac{\mathcal{A}(BH)}{4G}\,, \tag{163}$$

which is simply the Bekenstein-Hawking entropy. This provides an interpretation of the Bekenstein-Hawking entropy of a black hole in AdS spacetime as the thermal entropy of the dual CFT state.

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