From Singularities, via Entropy to Cosmology

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Abstract

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The problem of finding reasonable moduli spaces for algebraic singularities, has a long history, starting with the Italian algebraic geometry school, and made famous by Zariski.
30 years ago, Gerhard Pfister and I published a Springer Lecture Note (1310), called "Local moduli and Singularities", which became the starting point for my interest in Non-commutative Algebraic Geometry, and Quantum theory. It lead to subjects like Non-commutative Deformation Theory, Algebraic Entropy, and Cosmology, treated in several papers and two books, in particular: Geometry of Time Spaces, World Scientific (2011).
In this talk I shall sketch the development of the ideas behind these subjects, and hopefully, be able to explain how the singularity U:(the 4-dimensional real affine algebra of) a point with a 3-dimensional tangent space, in affine 3-space, lead to a rather interesting "Toy Model" for a Big Bang-scenario in Cosmology.

Abstract

To make this palatable, I shall first treat some seemingly trivial examples, showing how homological algebra and non-commutative algebraic geometry may give new insight in the classical theory of algebraic plane curves. Then I shall recall the basics of deformation theory, classical as well as noncommutative, before defining the notion of Entropy for singularities, as well as for general algebraic objects with algebraic versal base-spaces. Then I shall focus on the seemingly innocent, special, singularity U. The rest will be a treatment of my "Toy Model" of the Universe, with the Big Bang created by U..

All comments are welcome, but participants cannot expect full proofs for the central results. A list of references and some texts/preprints follows.

Algebraic Plane Curves

Let us start with the notion of plane curves and their singularities, given by a polynomial, $f \in k[x_1, x_2]$, and put,

$$A := k[x_1, x_2]/(f)$$

Let C be the affine plane curve $Simp_1(A) = Spec(A^{com})$, and fix a k-point $c \in C$, corresponding to a maximal ideal $\mathfrak{m} \subset A$. The Zariski-tangent space for C at c is defined as,

$$T_c := (\mathfrak{m}/\mathfrak{m}^2)* = Der_k(A, End_k(k(c))), k(c) := A/\mathfrak{m}$$

such that for the cusp, $f = x_1^3 - x_2^2$ the generic point has a tangent space of dimension 1, but $c = (0,0)$ is "Singular", and has a tangent space that is too big, $T_{(0,0)} = k^2$.
Recall that for any Associative Algebra A, and any two representation (here considered as right modules), $\rho_i : A \longrightarrow End_k(V_i), i = 1, 2$,

$$Ext^1_A(V_1, V_2) := Der_k(A, Hom_k(V_1, V_2)/Triv,$$

so, in the example above, $T_c = Ext^1_A(k(c), k(c))$.

Non-commutative Algebraic Plane Curves

Let $f \in k < x_1, x_2 >$ and put,

$$A := k < x_1, x_2 > /(F)$$

where $F = f - t[x_1, x_2]$. Put $C := Simp_1(A) = Spec(A^{com})$, then there exist an algebraic relation,

$$\mathfrak{R} \subset C \times C$$

such that for two points, $c_1, c_2 \in C$, we have,

$$(c_1, c_2) \in \mathfrak{R} \iff Ext^1_A(k(c_1), k(c_2)) \neq 0$$

Moreover, the projections from $\mathfrak{R}$ to C are dominant. This turns out to be a reason for considering any relation $(c_1, c_2) \in \mathfrak{R}$ as a tangent between the two points c_1 and c_2, or, simply considering,

$$Ext^1_A(k(c_1), k(c_2))$$

as the space, T_{c_1,c_2}, of Tangents from c_1 to c_2.

Non-commutative Elliptic Curves

Let $f = y^2 - x^3 - g_2x^2 - g_3x^3 \in k < x_1, x_2 >$ be an elliiptic curve, and put,

$$A := k < x_1, x_2 > /(F)$$

where $F = f - t[x_1, x_2]$. Let C be the affine plane curve $Simp_1(A) = Spec(A^{com})$, where A^{com} is the commutativization of A. Then the algebraic relation above, defined by,

$$(c_1, c_2) \in \mathfrak{R} \iff Ext^1_A(k(c_1), k(c_2)) \neq 0$$

is simply given by the Addition on the curve. In fact, let $c \in C$, and let the line l, with slope t, through the point $-c$, cut the curve in two other points c_1, c_2, then $(c_1 + c_2) = c$, and $(c, c_i) \in \mathfrak{R}, i = 1, 2$.

Deformations of Associative Algebras

We fix a field k. All algebras occurring, will be associative k-algebras. A Deformation of an algebra A Parametrized by the (maybe non-commutative) k-algebra, R, is a flat k-algebra homomorphism, $\mu : R \to A_R$, such that there is a "point", i.e. a homomorphism, $\rho : R \to k$, for which $k \otimes_R A_R \simeq A$. Two such deformations $\mu_i : R \to A_R$ are isomorphic if there exists a R-isomorphism $\psi : A_R \simeq A_R$, reducing to the identity of A over k, with μ_2 equal to μ_1 composed with ψ.
Define the Deformation Functor,

$$Def_A : \underline{a}_1 \to \underline{Sets}$$

where $\underline{a}_1$ is the category of Artinian 1-pointed k-algebras $R \to k$, by,

$$Def_A(R) := \{R \to A_R, deformation\}/Isomorphisms.$$

Formal Moduli and Pro-representig (Modular) Substratum

For any associative k-algebra A, with finite dimensional cohomology groups $A^1(k, A : A)$, defined in the next frame, there is a formal moduli, i.e. a complete local k-algebra, $\mathbf{H}(A)$, and a deformation, a Versal Family $\mu : \mathbf{H}(A) \to \mathbf{A}$, of associative algebras, such that for any other deformation $R \to A_R$, with R a finite dimensional local k-algebra, there is a diagram,

$$\begin{array}{ccc} \mu : \mathbf{H}(A) & \longrightarrow & \mathbf{A} \\ \downarrow & & \downarrow \\ R & \longrightarrow & A_R \end{array}$$

with $A_R \simeq R \otimes_{\mathbf{H}(A)} \mathbf{A}$. An optimal quotient $\mathbf{H}(A) \to \mathbf{H}_0(A)$, for which the composition $Hom_k(\mathbf{H}_0(A), -) \to Hom_k(\mathbf{H}(A), -) \to Def_A(-)$, is injective, is called a Prorepresenting Stratum (of $\mathbf{H}(A)$).

The tangent space $T^\star$

The tangent space of $\mathbf{H}(A)$, i.e. the dual k vectorspace of $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of $\mathbf{H}(A)$, corresponding to A, is calculated as:

$$T^\star = A^1(k, A; A) = Hom_F(ker(\rho), A)/Der$$

where, $\rho : F \to A$, is a surjective homomorphism of a free k-algebra F onto the given algebra A, Hom_F is the set of F-bilinear maps, and $Der \subset Hom_F$, denotes the restrictions to $ker(\rho)$, of the derivations $Der_k(F, A)$. (These A^i-groups are essentially the Hochschild cohomology groups.)
$A^0(k, A; A) = Der_k(A)$ is a Lie algebra, and there is a canonical action,

$$Der_k(A) \to End_k(A^1(k, A; A))$$

that may be "easily computed". See LP (2.11).

The universal obstruction morphism

Let $T^i = k[[A^i(k, A; A)^*]], i = 1, 2$ be the formal power series algebra in a basis of the dual vector space $A^i(k, A; A)^*$ (here assumed to be of finite dimension). Then there is a morphism of k-algebras,

$$o_X : T^2 \to T^1$$

such that,

$$\mathbf{H}(A) \simeq T^1 \otimes_{T^2} k.$$

Moreover o_X can be explicitly calculated using the Massy product structure of the cohomology $A^*(k, A; A)$.
NB! All this holds, in a vast generality, for most algebraic objects, X with finite dimensional cohomology $A^1(k, X; O_X)$.

The Kodaira-Spencer map

Now assume that $\mu : \mathbf{H}(A) \to \mathbf{A}$ has an algebraisation, $\pi : \tilde{H}_A \to \tilde{A}$.
Then there is a morphism

$$g : Der_k(\tilde{H}_A) \to A^1(\tilde{H}_A, \tilde{A}; \tilde{A}),$$

called the Gauss map, or more commonly, the Kodaira-Spencer map. The kernel, $V := ker(g)$ is a sub Lie algebra of $Der_k(\tilde{H})$, and the prorepresenting stratum, $\underline{\tilde{H}}_0 \subset \underline{\tilde{H}}$ is the fixed subset of the action of V.
Equivalently, $\tilde{H} \to \tilde{H}_0$ is the quotient killing the action of V.
In the next few frames, we shall assume that there is a, not only an algebraisation, but a versal one, i.e. a flat homomorphism of k-algebras, $H \to \tilde{A}$, such that for all (points of $\underline{H}$), $\rho_t : H \to k$, there is a surjection from $\hat{H}_t$ to $\mathbf{H}(\tilde{A} \otimes_H k(t))$. Put $A(t) := \tilde{A} \otimes_H k(t))$ and

$$\tau(t) := dim_k A^1(k, A(t), A(t))$$

Entropy

Consider an algebraic geometric object X, and let $aut(X)$ be the Lie algebra of automorphisms of X. The sub-Lie algebra $aut_0(X)$ that lifts to automorphisms of the formal moduli of X, is a Lie ideal. Put $\mathfrak{a}(X) := aut(X)/aut_0(X)$, then if $X(t)$ is a deformation of some X along a parameter t, we find $\mu X(t) := dim_k \mathfrak{a}(X(t)) \leq dim_k \mathfrak{a}(X) := \mu(X)$. One may phrase this saying that an object X can never gain *information* when deformed. Moreover, deformation is, obviously, not always a reversible process, so information can get lost. This measure of information losses, is related, as we shall see, to the notion of gain of entropy (en-ergy and tropos=transform) coined by Clausius (1865) and generalised by Boltzmann and Shannon.

The Classical Commutative Case

In (La-Pf), studying moduli problems of singularities A, in (classical) algebraic geometry, we were led to consider the notion of Modular Suite. This is a canonical partition $\{M_\tau\}$, of the versal base space, $\underline{H}$, of the deformation functor of an algebra A. The different *rooms*, $M_\tau = \{t \in \underline{H} | \tau(t) = \tau\}$, are filtered into subsets of $\underline{H}$, along which the Lie algebra $\mathfrak{a}(A) := aut(A)/aut_0(A)$ deforms as Lie-algebras, and therefore conserves its dimension. Consequently we obtain a finer Moduli Suite, $M_{\tau,\mu} = \{t \in \underline{H} | \tau(A(t)) = \tau, \mu(A(t)) = \mu\}$,

Working with Thermodynamics, it occurred to me that the notion of entropy has an interesting parallel in deformation theory. In fact I have proposed the following,

In general fix an algebraic object X, and let $X(\underline{t})$ corresponds to the point $\underline{t} \in M_{\tau,\mu}$, then we shall term Entropy, of the *state* $\underline{t}$, the integer,

$$S(\underline{t}) := dim(M_{\tau,\mu})$$

Results for Singularities

This functions well, in the case of singularities, but recall that a singularity is, by definition, a formal object, of the form, $\hat{A}_{\mathfrak{m}}$, $\mathfrak{m}$ a maximal ideal. In this classical situation, assuming that the field is algebraically closed, and that $\underline{H}$ is of finite Krull dimension, the modular suite $\{M_{\tau,\mu}\}$ is finite, with an inner room, the modular substratum and an ambient (open) maximal entropy stratum. But the structure of the modular suite may be very complex, even for simple singularities A, see the example of the quasi homogenous plane curve singularity $x_1^5 + x_2^{11}$, in (La-Pf). It is also clear that for any *reasonal algebraic dynamics* in $\underline{H}$, the entropy will always stay or grow, see again (La-Pf). To be able to construct situations where the entropy is lowered, or the information goes up, we must leave classical algebraic geometry, and venture into non-commutative algebraic geometry.

Noncommutative Deformations of Families of Representations

Given a k-algebra A, and a finite family $\mathfrak{V}$, of representations,

$$\rho_i : A \to End(V_i)$$

and consider the deformation functor,

$$Def_{\mathfrak{V}} : \underline{a}_r \to \underline{Sets}$$

where $\underline{a}_r$ is the category of Artinian r-pointed k-algebras. An object of $\underline{a}_r$ is a diagram, $R : k^r \to^i R \to^\pi k^r$, (with composition $i\pi = id$) and where $Def_{\mathfrak{V}}$ is defined by,

$$Def_{\mathfrak{V}}(R) = \{\phi : A \to (R_{i,j} \otimes Hom_k(V_i, V_j)\}/Iso$$

ϕ commuting with the left action of $R = (R_{i,j})$ and reducing, via π to the (right) A-module structure of the sum of $\rho_i : A \to End_k(V_i)$.

The Universal Obstruction Morphism

As in the case of algebras, therer exist a morphism of k^r-algebras,

$$o_X : T^2 \to T^1$$

such that,

$$H := H(\mathfrak{V}) \simeq T^1 \otimes_{T^2} k^r.$$

is a formal moduli for the family $\mathfrak{V}$. Moreover, there is an up to isomorphisms unique morphism of algebra,

$$\eta : A \to (H_{i,j} \otimes Hom_k(V_i, V_j)) =: O(\mathfrak{V})$$

A very important result in noncommutative algebraic geometry, is the Generalised Bernside Theorem: If A is Artinian and $\mathfrak{V}$ consists of all the simple modules of A, then η is an isomorphisme.

Deformations of a Thick Point in 3-space, as Associative Algebras

- $A = k[x_1, x_2, x_3]$ is the commutative coordinate ring of the affine 3-space.
- $U = A/(\underline{x})^2$ is, geometrically, a thick point in affine 3-space, but
- U is also a quotient of the free associative k-algebra, $F = k < x_1, x_2, x_3 >$,
- Let $\rho : F \rightarrow U$ be the quotient map, then the kernel, $ker(\rho) = (x_i x_j),\ i, j = 1, 2, 3$

A deformation of U parametrized by the (commutative) k-algebra, B, is a flat k-algebra homomorphism,
$B \rightarrow B < x_1, x_2, x_3 > /(x_i x_j + \sum b^l_{i,j} x_l + b^0_{i,j})$
It is easy to compute the tangent space of the formal moduli of U,

$$dim_k A^1(k, U; U) = dim_k Hom_F(ker(\rho), U)/Der = 27.$$

The restricted formal moduli of U

Assume $k = \mathbf{R}$, and pick any two real 3-vectors,

$$\overline{o} := (o_1, o_2, o_3), \overline{p} := (p_1, p_2, p_3).$$

The bi-linear homomorphisms,

$$\kappa : ker(\rho) = (x_i x_j) \to U,\ \kappa(x_i x_j) := o_i x_j + x_i p_j$$

represents linearly independent elements in $A^1(k, U; U)$, and span a subspace of the tangent space of the formal moduli of U generating a quotient of the versal family,

$$\begin{array}{ccc} \mu : \mathbf{H}(U) & \longrightarrow & \tilde{\mathbf{U}} \\ \downarrow & & \downarrow \\ H(U) & \longrightarrow & \mathbf{U} \end{array}$$

The restricted versal family

- Let $\underline{o} := (o_1, o_2, o_3)$, and $\underline{p} := (p_1, p_2, p_3)$ be sets of independent coordinates, and let,
- $H(U) =: H = k[o_1, o_2, o_3, p_1, p_2, p_3]$
- Put, $\underline{H} := Spec(H) = \mathbf{A}^3 \times \mathbf{A}^3$, and we have got a deformation of U,

$$\tilde{\rho} : H \to \mathbf{U} := H < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - x_i p_j + o_i p_j)$$

parametrized by the 6-dimensional scheme $\underline{H}$.

- Let $o := (o_1, o_2, o_3)$, and $p := (p_1, p_2, p_3)$ be two vectors, then $U(o, p) := k < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - x_i p_j + o_i p_j)$
- NB Put, $B = k[t], b^l_{i,j} = \epsilon_{i,j,l} t,\ b^0_{i,j} = \delta_{i,j} t^2$ then the deformation of U along t for $t \neq 0$ is constant, and equal to the Quaternions.

The Moduli Suite of U

- The modular substratum of the versal base space of U is easily seen to be $\underline{\Delta} = \{(\underline{o}, \underline{p}) I \underline{o} = \underline{p}\}$
- Our restricted family, $\underline{H} - \underline{\Delta} \subset \mathbf{A}^3 \times \mathbf{A}^3$, is another room. And here the $GL(3)$ operates such that any $g \in GL(3)$ induces an isomorphism $U(\underline{o}, \underline{p}) \simeq U(g(\underline{o}), g(\underline{p}))$. Moreover this is, in a sense, the maximal entropic subset of the modular suite.
- NB There are rooms of less entropy, like for that containing the exterior algebra in dimension 2. And then there is a room consisting of a unique singular point, given by the deformation $B = k[t], b^l_{i,j} = \epsilon_{i,j,l} t,\ b^0_{i,j} = \delta_{i,j} t^2$ and the Quaternions.

$\mathbf{U}$ extends to $\mathbf{H} := Hilb^2(\mathbf{A}^3)$

No 1

For every $(o,p) \in \mathbf{A}^3 \times \mathbf{A}^3 = \underline{H}$, there is an associative 4-dimensional k-algebra $U(o,p)$, the fibre of $\mathbf{U}$ at (o,p). Moreover, if $o = p$, then $U(o,p) \simeq U$. We obtain the extended diagram,

$$\begin{array}{ccccccc}
\underline{\tilde{H}} & \longrightarrow & \underline{H} & \longleftarrow & \underline{\Delta} & \longleftarrow & \underline{\tilde{\Delta}} \subset \underline{\tilde{H}} \\
\downarrow {\scriptstyle Z_2} & & \uparrow & & \uparrow & & \\
\mathbf{H} & \longleftarrow & \mathbf{U} & \longleftarrow & U & &
\end{array}$$

This follows from the following elementary relations: For $(o,p) \in \underline{H}$, for every $c \in \mathbf{A}^3$ and for any non-zero $\kappa \in k$, we have,

- $U(\kappa o, \kappa p) \simeq U(o,p)$
- $U(o,p) \simeq U(o-c, p-c)$
- $U(-p,-o) \simeq U(o,p)$

A Universal Gauge Group

No 1

Consider the bundle of Lie algebras, defined on **H** by,

- $\mathfrak{g} := Der_{\mathbf{H}}(\mathbf{U})$
- $\mathfrak{g}(o,p) = Der_{k(o,p)}(U(o,p)), (o,p) \in \underline{H}$.

Any element $\delta \in \mathfrak{g}(o,p)$ must have the form,
$\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3$. Consider the 4-vectors,
$\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3),\ \overline{o} = (1, o_1, o_2, o_3),\ \overline{p} = (1, p_1, p_2, p_3)$

Theorem

- $\delta \in \mathfrak{g}(o,p)$ *if and only if* $\delta_i.\overline{o} = \delta_i.\overline{p} = 0$,
- *If* $o \neq p$, *then,* $\mathfrak{g}(o,p) \simeq \begin{pmatrix} 0 & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix}$
- $rad(\mathfrak{g}) = \{u, r_1, r_2\},\ \mathfrak{g}/rad \simeq \mathfrak{sl}(2) = \{h, e, f\} \subset \mathfrak{g}$
- $h \in \mathfrak{h} \subset \mathfrak{g}$*; the generator of the Cartan algebra.*

Canonical Basis for Θ_H

No 3

Notice also that, in this case, the unique 0 (resp light)-velocity tangent line at the point $\underline{t}_0 = (o, p)$, $o = (0,0,0), p = (1,0,0)$, killed by $\mathfrak{g}$, is represented by,

- $d_3 := ((1,0,0),(1,0,0))$, the unique 0-velocity, resp.
- $c_3 := ((1,0,0),(-1,0,0))$, the unique light-velocity.

Let, $d_1 := ((0,1,,0),(0,1,0))$, $d_2 := ((0,0,1),(0,0,1))$ and let, $c_1 := ((0,1,,0),(0,-1,0))$, $c_2 := ((0,0,1),(0,0,-1))$,.

- Then $\{c_1, c_2, c_3, d_1, d_2, d_3\}$ is a basis for the tangent space $\Theta_{\underline{t}_0}$, $\{c_1, c_2, c_3\}$ for $\tilde{c}_{\underline{t}_0}$, and $\{d_1, d_2, d_3\}$ for $\tilde{\Delta}_{\underline{t}_0}$.

The Toy Model

No 1

My favourite "Toy Model", of General Relativity, and Quantum Theory is the philosophically reasonable (?) Physical Model, of an Observer and an Observed in 3-dimensional space, mathematically modelled by the Hilbert scheme $\mathbf{H}$ of length 2 sub-schemes in $\mathbf{A}^3$. Consider the diagonal, $\underline{\Delta} \subset \mathbf{A}^3 \times \mathbf{A}^3 = \underline{H}$, and let $\underline{\tilde{H}}$ be the blow up of $\underline{H}$ in $\underline{\Delta}$. We find a diagram,

$$\begin{array}{ccc} \underline{\tilde{H}} & \longrightarrow & \underline{H} \supset \underline{\Delta} \longleftarrow \underline{\tilde{\Delta}} \subset \underline{\tilde{H}} \\ \downarrow{\scriptstyle Z_2} & & \\ \mathbf{H} & & \end{array}$$

where $\mathbf{H} = Hilb^2(\mathbf{A}^3) = \underline{\tilde{H}}/Z_2$. Call he generator, P of Z_2, the Parity operator, and fix, coordinates for $\mathbf{H}$: $(\underline{\lambda}, \pm\underline{\omega}, \rho)$, with $\underline{\lambda} \in \underline{\Delta}$, $\underline{\omega}$ a spherical coordinate of the blow-up of $\underline{\Delta}$ in $\underline{H}$, and ρ the length from $\underline{\Delta}$ along the line defined by $\underline{\omega}$.

Coordinates of $\underline{\tilde{H}}$

No 2

This gives also a convenient parametrisation of $\underline{\tilde{H}}$. Consider, as above, for each $\underline{t} \in \underline{\tilde{H}}$ the length ρ, in the Euclidean space, $\mathbf{E}^3$, of the vector $1/2(p-o)$. Given a point $\underline{\lambda} = (c, c) \in \underline{\Delta}$, and a point $\omega \in E(\underline{\lambda}) = \pi^{-1}(\underline{\lambda})$, of the fibre of,

$$\pi : \underline{\tilde{H}} \to \underline{H},$$

at the point $\underline{\lambda}$. Since $E(\underline{\lambda})$ is isomorphic to S^2, parametrized by $\underline{\omega} = (\phi, \theta)$, any element $\underline{t} := (o, p)$ of $\underline{\tilde{H}}$ is uniquely determined in terms of the triple $\underline{t} = (\underline{\lambda}, \underline{\omega}, \rho)$, where $\underline{\lambda} = c(o, p) = (c, c)$, the centre of (o, p), and such that $\underline{\omega}$ is defined by the directed line $\underline{op}$, and the action of θ keeps ω fixed. Here $\rho \geq 0$, and notice that, at the exceptional fibre, i.e. for $\rho = 0$, the momentum corresponding to $d\rho$ is not defined.

Metrics or Time

No 3

We shall consider metrics, (and therefore also the notions of Time) of the Model $\underline{\tilde{H}}$, of the form,

$$g = h_\rho(\underline{\lambda}, \underline{\omega}, \rho)d\rho^2 + h_{\underline{\omega}}(\underline{\lambda}, \underline{\omega}, \rho)d\underline{\omega}^2 + h_{\underline{\lambda}}(\underline{\lambda}, \underline{\omega}, \rho)d\underline{\lambda}^2,$$

where $d\underline{\omega}^2$ is the natural metric in $S^2 = E(\underline{\lambda})$. Pick the simplified space, in which $\underline{\omega}$ is reduced to the angle ϕ, and the coordinates $\underline{\lambda}$ is reduced to one parameter $\lambda = |\underline{\lambda}|$, and the metric is,

$$g = (\frac{\rho - h(\lambda)}{\rho})^2 d\rho^2 + (\rho - h(\lambda))^2 d\phi^2 + \kappa(\lambda)d\lambda^2,$$

This correspond to considering the sub-universe of $\mathbf{M}(B)$, parametrized by (λ, ϕ, ρ).

Computing the geodesics (the Force Law), taking into account that Time is the metric, we find,

$$\frac{d^2\lambda}{dt^2} = -(\frac{\rho - h(\lambda)}{\rho})(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})^2 - (\rho - h(\lambda))(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\phi}{dt})^2 + (\frac{dlog(\kappa)}{d\lambda})(\frac{d\lambda}{dt})^2$$

Big Bang Model

No 5

Moreover we find,

$$\frac{d^2\rho}{dt^2} = -(\frac{h(\lambda)}{\rho(\rho - h(\lambda))})(\frac{d\rho}{dt})^2 + (\frac{2}{(\rho - h(\lambda))})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})(\frac{d\lambda}{dt}) + (\frac{\rho^2}{(\rho - h(\lambda))})(\frac{d\phi}{dt})^2,$$

$$\frac{d^2\phi}{dt^2} = -2/(\rho - h(\lambda))\frac{d\rho}{dt}\frac{d\phi}{dt} + 2/(\rho - h(\lambda))(\frac{dh}{d\lambda})(\frac{d\phi}{dt})(\frac{d\lambda}{dt})$$

where t, is the time parameter of the model. From these formulas we see that the Gravitation is expanding inside the Horizon and contracting outside. Conservation of mass implies, $h(\underline{\lambda}) = h_0/\lambda$. From this follows that the Horizon at the BB, i.e. for $\lambda = 0$, is all of space. Interpreting λ as Cosmological time we find a striking cosmological model, complete with Inflation and Hubble formulas, $v = r/t$ and $v/\sqrt{1 - v^2} = r/\lambda$.

Now, let $h(\underline{\lambda}) = h$ be constant, then the geodesics have the equations,

$$\frac{d^2\rho}{dt^2} = -(\frac{h}{\rho(\rho-h)})(\frac{d\rho}{dt})^2 + (\frac{\rho^2}{(\rho-h)})(\frac{d\phi}{dt})^2,$$
$$\frac{d^2\phi}{dt^2} = -2/(\rho-h)\frac{d\rho}{dt}\frac{d\phi}{dt}, \quad Kepler's\ 2.Law.$$
$$\frac{d^2\lambda}{dt^2} = 0,$$

The definition of time gives us,

$$\rho^{-2}(\frac{d\rho}{dt})^2 = (\rho-h)^{-2}K^2 - (\frac{d\phi}{dt})^2.$$

where, $K^2 = (1-(\frac{d\lambda}{dt})^2)$, is the kinetic energy.

Kepler and Newton

No 7

Put this into the first equation above, and obtain,

$$\frac{d^2\rho}{dt^2} = -hK^2(\frac{\rho}{\rho - h})\frac{1}{(\rho - h)^2} + (\frac{\rho + h}{\rho - h})\rho(\frac{d\phi}{dt})^2.$$

Assume now $r := \rho - h \approx \rho$, we find,

$$\frac{d^2 r}{dt^2} = -\frac{hK^2}{r^2} + r(\frac{d\phi}{dt})^2, \ \ Kepler's\ 1.Law.$$

The constant h, the radius of the exceptional fibre, is thus also related to mass. Recall that the Schwarzschild radius, the Einstein equivalent to h, is assumed to be,

$$r_s = 2GM/c^2,$$

where, G = Newton's gravitational constant, M = mass, c = speed of light, which here, of course, is put equal to 1.

Spin and Isospin
No 1

Denote by $\tilde{\Delta} \subset \tilde{\Theta}_{\underline{\tilde{H}}}$ the 3-dimensional distribution, generated by the translations $\{(o,p) \to (o+c, p+c),\ c \in \mathbf{A}^3\}$, and complexify all bundles. Introduce a Riemanian metric g on the space $\mathbf{H}$, and see that we have the following structures uniquely defined,

- $\tilde{\Delta}_{\mathbf{C}}$, with the action of $\mathfrak{su}(3)/\mathfrak{so}(3)$
- $\Theta_{\underline{\tilde{H}}} \simeq B_o \oplus B_p \oplus A_{o,p} \simeq \tilde{\Delta} \oplus \tilde{c}$; resp. 0-and light-velocities.
- There exists a canonical action of $\mathfrak{g}_{\mathbf{C}}$ on $\Theta_{\tilde{H},\mathbf{C}}$, respecting the decomposition, such that,
- $\mathfrak{g}(o,p)$ acts on the tangent space $T_{\mathbf{H},(o,p)} = T_{\mathbf{A}^3,o} \times T_{\mathbf{A}^3,p}$ killing the vector $p-o$, in both factors.

The canonical action of these principal Lie bundles on the complexified tangent bundle $\Theta_{\mathbf{H}}$ "determins all Fields".

Action of the Gauge group $\mathfrak{g}$

No 2

The generators, $h, e, f \in \mathfrak{sl}(2) \subset \mathfrak{g}$ act, in the above basis, like,

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Action of the Gauge group $\mathfrak{g}$
No 3

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The generators, $u, r_1, r_2 \in rad(\mathfrak{g})$ act, in the above basis, like,

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Action of the Gauge group $\mathfrak{g}$

No 4

$$r_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

u, r_1, r_2 and Weak Interaction

Action of the Gauge group $\mathfrak{s}u(3)$

No 1

We observe here that the rotations (spin) around the 3 different axes in $\tilde{c}$, respectively in $\tilde{\Delta}$, are given by, $(e-f),(e-r_1),(f-r_2)$, about the axe defined by $(op)=c_3,c_2,c_1$, respectively $(op)=d_3,d_2,d_1$,
Recall that $\tilde{\Delta}\subset\Theta_{\underline{\tilde{H}}}$, is the sub-bundle defined, at the point $\underline{t}$ as the space of tangents of the form (ξ,ξ). Given a metric on $\underline{\tilde{H}}$, we may look at the action of $\mathfrak{su}(3)$ on $\tilde{\Delta}\otimes\mathbf{C}$. Knowing that $\Theta=\tilde{c}\oplus\tilde{\Delta}$, $\mathfrak{su}(3)$ acts in the obvious way on the lower right corner, like

$$\gamma=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&*&*&*\\0&0&0&*&*&*\\0&0&0&*&*&*\end{pmatrix}.$$

Action of the Gauge group $\mathfrak{s}u(3)/\mathfrak{so}(3)$

No 2

Since the 0-velocity direction defined at (o, p), is $(o - p, o - p)$, which here is d_3, we may in an essential unique way decompose the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{su}(3)$, into the Cartan subalgebra $\mathfrak{h}_1$, for the $\mathfrak{su}(2)$-component leaving δ_3 invariant, and the part $\mathfrak{h}_2 \subset \mathfrak{h}$ perpendicular, in the Killing metric, to $\mathfrak{h}_1$.

$$\mathfrak{h}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathfrak{h}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2/3 \end{pmatrix},$$

Classically one denotes the other 6 base elements of $\mathfrak{s}u(3) \otimes \mathbf{C}$, as, $\mathfrak{e}^i_{\pm}, i = 1, 2, 3$.

Action of the Gauge Group

No 3

The restriction to $\tilde{\Delta}$ of these operators, in the basis $\{d_1, d_2, d_3\}$ are given by,

$$\mathfrak{e}_+^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{e}_+^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{e}_+^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and their duals,

$$\mathfrak{e}_-^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{e}_-^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathfrak{e}_-^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Commutators in $\mathfrak{su}(3)$

No 4

We need the commutators, $[\mathfrak{h}_1, \mathfrak{h}_2] = 0$, and,

$$[\mathfrak{h}_1, \mathfrak{e}^1_\pm] = \pm\mathfrak{e}^i_\pm, [\mathfrak{h}_1, \mathfrak{e}^2_\pm] = \mp 1/2\mathfrak{e}^2_\pm, [\mathfrak{h}_1, \mathfrak{e}^3_\pm] = \pm 1/2\mathfrak{e}^3_\pm$$

together with the following ones,

$$[\mathfrak{h}_2, \mathfrak{e}^1_\pm] = 0, [\mathfrak{h}_2, \mathfrak{e}^2_\pm] = \pm\mathfrak{e}^2_\pm, [\mathfrak{h}_2, \mathfrak{e}^3_\pm] = \pm\mathfrak{e}^3_\pm$$

Notice, for later use, that the only non-zero products of the $\mathfrak{e}^i_\pm$ are,

$$\mathfrak{e}^1_+ \cdot \mathfrak{e}^2_+ = \mathfrak{e}^3_+, \mathfrak{e}^2_- \cdot \mathfrak{e}^1_- = \mathfrak{e}^3_-.$$

Elementary Particles

No 1

Together these formulas show that the quotients of of the $\mathfrak{g}*$-representation $\Theta_{\tilde{H}}$ are the following,

- $\tilde{c}$ and therefore also the *photon* $\{c_1, c_2\}$ and a singleton, $\{c_3\}$, both simple.
- $\tilde{\Delta}$ and therefore the *electron* $\{d_1, d_2\}$ and a singleton, $\{d_3\}$, both simple.
- Weyl spinors, B_o, B_p, and Dirac spinors, $B_o \oplus B_p$

The non-trivial simple quotients of of the $\mathfrak{su}(3)$-representation $\Theta_{\tilde{H}}$ are reduced to,

- The quarks, $\tilde{\Delta}$

Photons

No 2

We observe that the action of $\phi \in u(1)$ is, $exp(\imath\phi \cdot (e-f))$, on the the (transverse) light-wave is given by,

$$A(\phi) = \begin{pmatrix} cos\phi & -sin\phi & 0 & 0 & 0 & 0 \\ sin\phi & cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is clear that this, together with the formulas above give good reasons to believe that there is a relation between this model, and the Standard Model (and so also to the 8-fold way of Gell-Mann). Moreover, here all ingredients are universally given by the information contained in the singularity U, the Big Bang, in my tapping. The choice of metric depends, however, on the nature of what I have called the Furniture of the model.

The Pauli Matrices

No 3

It is now easy to see that the the Pauli matrices are found as follows,

$$\sigma^1 = e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma^2 = -ie + if = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$\sigma^3 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Moreover, the parity operator P, the generator γ of the symmetry group $\mathbf{Z}_2$, operating on $\underline{\tilde{H}}$, acts on $\tilde{c}$, as multiplication by (-1), see WS. Therefore, it maps the basis $\{(c_1 + d_1), (c_2 + d_2)\}$ into $\{(-c_1 + d_1), (-c_2 + d_2)\}$. Consequently, we have an isomorphism,

$$P : B_o \to B_p,$$

The Dirac Matrices

No 4

This shows that it is meaningful to consider the representations given by the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \ k = 1, 2, 3.$$

as well as the new operators,

$$\gamma^{k+3} = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}, \ k = 1, 2, 3,$$

acting on, $B_o \oplus B_p$, such that,

$$\forall p \neq q, \ \gamma^p\gamma^q = -\gamma^q\gamma^p, \gamma^p\gamma^p = 1, \ p, q = 1, 2, 3, 4, 5, 6.$$

Chirality

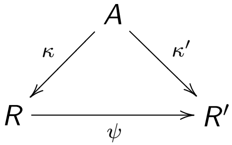
No 1

Chirality, in the physicists language, is explained as follows. The morphism P, extended to $B_o \oplus B_p$, in the basis chosen above, is given by the matrix,

$$P = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix},$$

which turns left-handedness to right-handedness, with respect to the direction (o, p), resp. (p, o).

Given an associative k-algebra A, denote by $A/k - \underline{alg}$ the category where the objects are homomorphisms of k-algebras $\kappa : A \to R$, and the morphisms, $\psi : \kappa \to \kappa'$ are commutative diagrams,



and consider the functor,

$$Der_k(A, -) : A/k - \underline{alg} \longrightarrow \underline{Sets}.$$

It is representable by a k-algebra-morphism, $\iota : A \longrightarrow Ph(A)$, with a universal family, i.e. a universal derivation,

$$d : A \longrightarrow Ph(A).$$

A universal derivation associated to an A-module

I No 2

Clearly we have the identities,

$$d_* : Der_k(A, A) = Mor_A(Ph(A), A),$$

and,

$$d^* : Der_k(A, Ph(A)) = End_A(Ph(A)),$$

the last one associating d to the identity endomorphism of Ph. Let now V be a right A-module, with structure morphism

$$\rho : A \to End_k(V).$$

We obtain another universal derivation,

$$c : A \longrightarrow Hom_k(V, V \otimes_A Ph(A)),$$

defined by, $c(a)(v) = v \otimes d(a)$.

The Kodaira-Spencer class

I No 3

Using the long exact sequence, of Hochschild cohomology,

$$0 \rightarrow Hom_A(V, V \otimes_A Ph(A)) \rightarrow Hom_k(V, V \otimes_A Ph(A)) \\ \xrightarrow{\iota} Der_k(A, Hom_k(V, V \otimes_A Ph(A))) \xrightarrow{\kappa} Ext^1_A(V, V \otimes_A Ph(A)) \rightarrow 0,$$

and,

$$c \in Der_k(A, Hom_k(V, V \otimes_A Ph(A))),$$

we obtain the non-commutative *Kodaira-Spencer class*,

$$c(V) := \kappa(c) \in Ext^1_A(V, V \otimes_A Ph(A)),$$

inducing, via the identity d_*, the *Kodaira-Spencer morphism*,

$$g : \Theta_A := Der_k(A, A) \longrightarrow Ext^1_A(V, V).$$

Hamiltonians and Connections

I No 4

Using again the long exact sequence, of Hochschild cohomology,

$$0 \to Hom_A(V,V) \to Hom_k(V,V)$$
$$\xrightarrow{\iota} Der_k(A, Hom_k(V,V)) \xrightarrow{\kappa} Ext^1_A(V,V) \to 0,$$

we prove,

Theorem

Let $\rho : A \to End_k(V)$, be an A-module, and let $\delta \in Der_k(A, Hom_k(V,V))$, map to 0 in $Ext^1_A(V,V)$, i.e. assume $\kappa(\delta) = 0$, then there exist an element, $Q_\delta \in Hom_k(V,V)$, the Hamiltonian, such that for all $a \in A$,

$$\rho(\delta(a)) = [Q_\delta, \tilde{\rho}(a)].$$

If V is a simple A-module, $ad(Q_\delta)$ is unique.

Chern Characters

I No 5

As is well known, in the commutative case, the Kodaira-Spencer class gives rise to a Chern character by putting,

$$ch^i(V) := 1/i!\ c^i(V) \in Ext^i_A(V, V \otimes_A Ph(A)),$$

and if $c(V) = 0$, the curvature $R(V)$ of ∇, induces a curvature class,

$$R_\nabla \in H^2(k, A; \Theta_A, End_A(V)).$$

The phase-space construction may be iterated. Given the k-algebra A we may form the sequence, $\{Ph^n(A)\}_{0\leq n}$, defined inductively by

$$Ph^0(A) = A,\ Ph^1(A) = Ph(A), ..., Ph^{n+1}(A) := Ph(Ph^n(A)).$$

Let $i_0^n : Ph^n(A) \to Ph^{n+1}(A)$ be the canonical imbedding, and let $d_n : Ph^n(A) \to Ph^{n+1}(A)$ be the corresponding derivation. Since the composition of i_0^n and the derivation d_{n+1} is a derivation $Ph^n(A) \to Ph^{n+2}(A)$, corresponding to the homomorphism,

$$Ph^n(A) \to^{i_0^n} Ph^{n+1}(A) \to^{i_0^{n+1}} Ph^{n+2}(A)$$

there exist by universality a homomorphism $i_1^{n+1} : Ph^{n+1}(A) \to Ph^{n+2}(A)$, such that,

$$i_0^n \circ i_1^{n+1} = i_0^n \circ i_0^{n+1}$$

and such that,

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}.$$

The Semi-Cosimplicial Structure of Ph*

I No 8

Clearly we may continue this process constructing new edge morphisms,

$$\{i_j^n : Ph^n(A) \to Ph^{n+1}(A)\}_{0 \leq j \leq n},$$

such that,

$$i_p^n \circ i_0^{n+1} = i_0^n \circ i_{p+1}^{n+1}$$

with the property,

$$d_n \circ i_{j+1}^{n+1} = i_j^n \circ d_{n+1}.$$

We find the following identities,

$$\begin{aligned} i_p^n i_q^{n+1} &= i_{q-1}^n i_p^{n+1}, \ p < q \\ i_p^n i_p^{n+1} &= i_p^n i_{p+1}^{n+1} \\ i_p^n i_q^{n+1} &= i_q^n i_{p+1}^{n+1}, \ q < p. \end{aligned}$$

The Semi-Cosimplicial Structure of Ph*

I No 9

To see this, compose with i_0^{n-1} and d_{n-1}, and use induction. Thus, the $Ph^*(A)$ is a semi-co-simplicial k-algebra with a co-section h_0, onto A. And it is easy to see that h_0 together with the corresponding co-sections $h_p : Ph^{p+1}(A) \to Ph^p(A)$, for $Ph^p(A)$ replacing A, form a trivialising homotopy for $Ph^*(A)$. Thus, we have,

$$H^n(Ph^*(A)) = 0, n \geq 0,$$

i.e.

- $Ph^*(A)$ is a cosimplicial resolution of the algebra A.

Therefore, for any object,

$$\kappa : A \to R \in A/k - \underline{alg}$$

the co-simplicial algebra above induces simplicial sets,

$$Mor_k(Ph^\star(A), R),\ Mor_A(Ph^\star(A), R),$$

and one should be interested in the homotopy.

Homotopy

I No 10

See also that this generalizes to a canonical functor,

$$Spec : (k - alg^{\Delta})^{op} \longrightarrow SPr(k)$$

where $(k - alg)^{\Delta}$ is the category of co-simplicial k-algebras, and $SPr(k)$ is the category of simplicial presheves on the category of k-schemes enriched by some Grothendieck topology.

As usual, the imbedding of the category of k-algebras in the category of co-simplicial algebras is defined simply by giving any k-algebra a constant co-simplicial structure. The fact that $Ph^{\star}(A)$ is a resolution of A, is therefore simply saying that,

$$Spec(Ph^{\star}((A)) \rightarrow Spec(A),$$

is a week equivalence in $SPr(k)$.

This might be a starting point for a theory of homotopy for (non-commutative) k-schemes.

Cohomology

I No 11

We may also consider, for any k-algebra R, the simplicial k-vectorspace,

$$Der_k(Ph^\star(A), R),$$

Consider this complex for $R = A$, i.e. $Mor_A(Ph^\star(A), A)$. Clearly, $Mor_A(Ph^{n+1}(A), A) = Der_k(Ph^n(A), A), n \geq 0$, and we have,

$$Mor_A(Ph^n(A), A) = \{\xi_0 \circ \xi_{i_1} \circ ... \circ \xi_{i_r} | 0 \leq i_l \leq i_{l+1} \leq n,$$

where $\xi_0 = id_A, \xi_i \in Der_k(A), i \geq 1\}$. Since Ph is a functor, and Ph^{*+1} is a co-simplicial resolution of A. we may apply this to any scheme X, given in terms of an affine covering $\mathbf{U}$, and obtain an algebraic homology (or cohomology), with converging spectral sequences,

$$E^1_{pq} = H_p(H_{\mathbf{U}}{}^{-q}(Der_k(Ph^\star(A), A))), E_{q,p} = H_{\mathbf{U}}^{-q}(H_p(Der_k(Ph^\star(A), A)))$$

If we, in $Mor_A(Ph^n(A), A)$, identify $\xi \sim \alpha\xi, \alpha \in k^*$, we obtain a rational cohomology with converging spectral sequences,

$$E_1^{pq} = H^p(H^q_{\mathbf{U}}(Mor_A(Ph^n(A), A), \mathbf{Q})), E_2^{q,p} = H^q_{\mathbf{U}}(H^p(Mor_A(Ph^n(A), A), \mathbf{Q}))$$

Relation to the de Rham complex I

Consider now the diagram,

$$
\begin{array}{ccccccccc}
A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_p^1} & Ph^2(A) & \xrightarrow{i_p^2} & Ph^3(A) & \xrightarrow{i_p^3} & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathfrak{m}_1^1 & \xrightarrow{i_p^1} & \mathfrak{m}_2^1 & \xrightarrow{i_p^2} & \mathfrak{m}_3^1 & \xrightarrow{i_p^3} &
\end{array}
$$

where, for each integer n, the symbol i_p^n, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $Ph^n(A)$ and $Ph^{n+1}(A)$ defined above, and where $\mathfrak{m}_n^1$ is the ideal of $Ph^n(A)$ generated by $im(d)$, which is the same as the ideal generated by the family, $\{i_p^{n-1}(i_p^{n-2}(...(i_p^1(d(A))...))\}$, for all possible p. And, inductively, let $\mathfrak{m}_n^m$ be the ideal generated by $\mathfrak{m}_n^1 \mathfrak{m}_n^{m-1}$.

Relation to the de Rham complex II

I No 13

We find an extended diagram,

$$
\begin{array}{ccccccccccc}
A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_p^1} & Ph^2(A) & \xrightarrow{i_p^2} & Ph^3(A) & \xrightarrow{i_p^3} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A & \xrightarrow{i_0^0} & A & \xrightarrow{i_p^1} & A & \xrightarrow{i_p^2} & A & \xrightarrow{i_p^3} & \cdots \\
& \searrow^{d} & & \searrow^{d} & & \searrow^{d} & & \searrow^{d} & \\
& & \mathfrak{m}_1^1/\mathfrak{m}_1^2 & \xrightarrow{i_p^1} & \mathfrak{m}_2^1/\mathfrak{m}_2^2 & \xrightarrow{i_p^2} & \mathfrak{m}_3^1/\mathfrak{m}_3^2 & \xrightarrow{i_p^3} & \cdots \\
& & & \searrow^{d} & & \searrow^{d} & & \searrow^{d} & \\
& & \color{red}{\mathfrak{m}_1^2/\mathfrak{m}_1^3} & \xrightarrow{i_p^1} & \mathfrak{m}_2^2/\mathfrak{m}_2^3 & \xrightarrow{i_p^2} & \mathfrak{m}_3^2/\mathfrak{m}_1^3 & \xrightarrow{i_p^3} & \cdots
\end{array}
$$

The diagonals are not necessarily complexes, but to kill all d^n, $n \geq 2$, it suffices to kill d^2, and for this it suffices to kill $d_1 d_0$, as one easily see, applying the edge homomorphisms to, $d_1(d_0(a))$ for all $a \in A$.

Curvature

I No 14

Definition

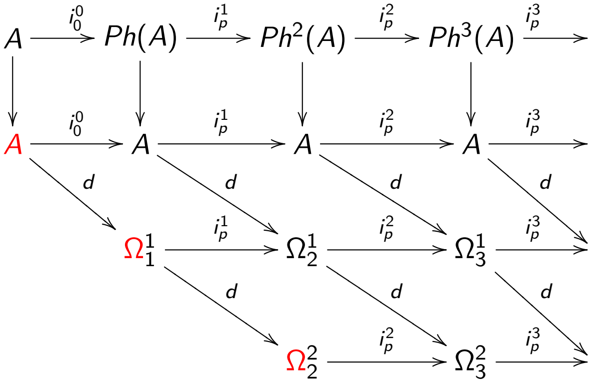
The curvature $R(A)$ of the associative k algebra, A, is the k-linear map composition of d_0 and d_1,

$$R(A) = d_0 d_1 : A \to \mathfrak{m}_2^2/\mathfrak{m}_2^3.$$

Now, kill the curvature $R(A)$, and all the terms under the first diagonal, beginning with $\mathfrak{m}_1^2/\mathfrak{m}_1^3$, together with all terms generated by the actions of the edge homomorphisms on these terms, and let, $\mathbf{\Omega}_n^m$ be the quotient of $\mathfrak{m}_n^m/\mathfrak{m}_n^{m+1}$, for $n \geq 0$. Clearly, $\Omega_n^0 = A$ for all $n \geq 0$, and we have got a graded semi co-simplicial A-module, with a k-differential d, such that $d^2 = 0$,

Generalized de Rham complex.

The diagram is now looking like,



It is therefore a graded complex, in two ways. First as a complex induced from the semi-cosimplicial structure, with differential of bidegree (1,0), and second, as complex with differential d, of bidegree (1,1).

The commutative case

I No 16

Consider now the complex,

$$A \rightarrow^d \Omega_1^1 \rightarrow^d \Omega_2^2 \rightarrow^d \Omega_3^3 \rightarrow^d \ldots$$

.

Theorem

Suppose A is commutative, then there is a natural morphism of complexes of A-modules,

$$\Omega_A^\star \subset \Omega_\star^\star,$$

with,

$$\Omega_A^n \simeq \Omega_n^n.$$

The proof

I No 17

Let, $a_i \in A, i = 1, ..., r$, and compute in $\Omega^r_\star$ the value of, $d^r(a_1 a_2 ... a_r)$. It is clear that this gives the formula,

$$\sum d_{i_1}(a_1) d_{i_2}(a_2)..d_{i_r}(a_r) = 0,$$

the sum being over all permutation $(i_1, i_2, ..., i_r)$ of $(0, 1, ..., r-1)$. Here we consider A as a subalgebra of $Ph^n(A)$ via the unique compositions of the $i^s_0 : Ph^s(A) \subset Ph^{s+1}(A)$. In particular, we have,

$$d_0(a_1)d_1(a_2) + d_1(a_1)d_0(a_2) = 0,$$

for all $a_1, a_2 \in A$. This relation and the relation $d_0(a)d_1(b) = d_1(b)d_0(a)$, which follows from commutativity, $d(a)b = bd(a)$, forcing the left and right A-action on Ω_A to be equal, immediately give us,
$d_0(a)d_1(b) = -d_0(b), d_1(a)$. It is now clear that the map that sends the element $da_1 \wedge da_2 \wedge ... \wedge da_r \in \Omega^r_A$ to $d_0(a_1)d_1(a_2)..d_{r-1}(a_r) \in \Omega^r_r$ is an isomorphism, and the rest should be clear.

Generalization to modules I

I No 18

Let now, V be an A-module, and assume $c(V) = 0$, and pick a connection, $\nabla \in Hom_k(V, V \otimes_A Ph(A))$ with $c = \iota(\nabla)$. This imply that for $a \in A$ and $v \in V$ we have $\nabla(va) = \nabla(v)a + v \otimes d_0(a)$. Composing ∇ with the cosection, $o : Ph(A) \to A$, corresponding to the 0-derivation of A, we therefore obtain an A-linear homomorphism $P : V \to V$, *a potential*. Since $i_0^0 : A \to Ph(A)$ is a section of o, we find a k-linear map,

$$\nabla_0 := \nabla - P : V \to V \otimes \mathfrak{m}_1^1$$

Using the property,

$$d_n \circ i_{j+1}^{n+1} = i_j^n \, d_{n+1},$$

it is easy to find well defined k-linear maps,

$$\nabla_1 : V \to V \otimes \Omega_2^2, \ \nabla_2 : V \to V \otimes \Omega_3^3, ..., \nabla_n : V \to V \otimes \Omega_{n+1}^{n+1} \ \forall n \geq 0,$$

given by ,

$$\nabla_{n+1} := \nabla_n \circ i_1^{n+1}, \ n \geq 0.$$

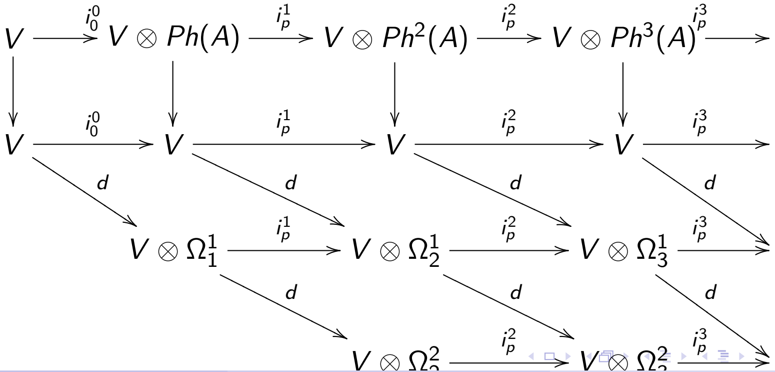
Generalization to modules

I No 19

Fix the connection ∇. For all, $v \in V, \omega \in \Omega_n^n$, the formula,

$$\nabla_n(v \otimes \omega) = \nabla_n(v)\omega + v \otimes d_n(\omega).$$

makes sense, and defines *derivations*, also called d. We obtain a situation just like above,



Generalization to modules

I No 20

In general, there are no reasons for these $d's$ to define complexes, and we shall make the following definition,

Definition

The curvature $R(V,\nabla)$ of the connection ∇ defined on the right k A-module V, is the k-linear map composition of d_0 and d_1,

$$R(V,\nabla) = d_0 d_1 : V \to V \otimes \Omega_2^2.$$

The following result is then easily proved,

Theorem

Suppose A is commutative, and let $\nabla : \Theta_A \to End_k(V)$ be the connection corresponding to ∇_0. Suppose moreover that the curvature R of ∇ is 0, then $R(V) = 0$, implying that $d^2 = 0$, and so the diagonals in the diagram above, are all complexes.

Dynamics and the Dirac Derivation

I No 21

Consider now the co-simplicial algebra,

$$A \rightarrow^{i_0^0} Ph(A) \rightarrow^{i_p^1} Ph^2(A) \rightarrow^{i_p^2} Ph^3(A) \rightarrow^{i_p^3} ....$$

where, for each integer n, the symbol i_p^n, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $Ph^n(A)$ and $Ph^{n+1}(A)$ defined above. The inductive (direct) limit,

$$Ph^\infty(A) = \varinjlim_{n \geq 0} \{Ph^n(A), i_j^n\}$$

comes with homomorphisms

$$i_n : Ph^n(A) \longrightarrow Ph^\infty(A), \text{satisfying} i_j^n \circ i_{n+1} = i_n, \quad j = 0, 1, \ldots, n$$

Moreover, the family of derivations, $\{d_n\}_{0 \leq n}$ defines a unique Dirac-derivation,

$$\delta : Ph^\infty(A) \longrightarrow Ph^\infty(A),$$

such that $i_n \circ \delta = d_n \circ i_{n+1}$.

Dynamics in Rep(A)

I No 25

Since $Ext^1_A(V,V)$ is the tangent space of the miniversal deformation space of V as an A-module, we see that the non-commutative space $Ph(A)$ also parametrizes the set of generalised momenta, i.e.the set of pairs of an A-module V, and a tangent vector of the formal moduli of V, at that point. Therefore the above implies that any representation, $\rho : Ph^\infty(A) \rightarrow End_k(V)$, corresponds to a family of $Ph^n(A)$-module-structures on V, for $n \geq 1$, i.e. to an A-module $V_0 := V$, an element $\xi_0 \in Ext^1_A(V,V)$, i.e. a tangent of the deformation functor of $V_0 := V$, as A-module, an element $\xi_1 \in Ext^1_{Ph(A)}(V,V)$, i.e. a tangent of the deformation functor of $V_1 := V$ as $Ph(A)$-module, an element $\xi_2 \in Ext^1_{Ph^2(A)}(V,V)$, i.e. a tangent of the deformation functor of $V_2 := V$ as $Ph^2(A)$-module, etc.

Dynamics in Rep(A)

I No 26

All this is just $\rho_0 : A \rightarrow End_k(V)$, considered as an A-module, together with a sequence $\{\xi_n\}, 0 \leq n$, of a tangent, or a *momentum*, ξ_0, an acceleration vector, ξ_1, and any number of higher order *momenta* ξ_n.

- Thus, specifying a $Ph^{\infty}(A)$-representation V, implies specifying a *formal curve* through v_0, the base-point, of the *miniversal deformation space* of the A-module V. Formally, this curve is given by the composition of the homomorphism $\epsilon(\tau) := exp(\tau\delta)$ and ρ.

Preparation and Time Evolution of Measurements

I No 27

A main problem of Science is the following:

- What do we have to know, to be able to predict?

It is impossible to prepare a physical situation such that a measurement, i.e. an object like ρ_0, is given by an infinite sequence $\{\xi_n\}$, of dynamical data. We shall have to be satisfied with a finite number of data, and normally with just the first one, i.e. the momentum ξ_0. This is the,

- Problem of Preparation for the Time Evolution of a representation ρ,

to be treated in the sequel.

Representations of $Ph^{\infty}(k[\underline{t}])$

Theorem

Given an r-dimensional $k[\underline{t}] := k[t_1, ..., t_d]$*-module, consisting of r points* $\{P_p = (\alpha_1^0(p), \alpha_2^0(p), ..., \alpha_d^0(p))\}_{p=1,...,r}$. *Assume given,* $\alpha_i^n(p) \in k, \; i = 1, ..., d, \; n \geq 0, \; p = 1, ..., r$, *and arbitrary coupling constants,* $\sigma_m(p, q) \in k$ *with* $\sigma_0 = 0$. *Put* $\alpha_i^n(p, q) = \alpha_i^n(p) - \alpha_i^n(q), \; i = 1, ..., d$, *and assume, for all* $n \geq 1$,

$$\sum_h \binom{n}{h} \sigma_{n-h}(p, q)(\alpha_i^h(p, q)\alpha_j^0(p, q) - \alpha_i^0(p, q)\alpha_j^h(p, q))$$

$$= \sum_{k,l,m,s} \frac{n!\sigma_{n-k-m}(p, s)\sigma_{k-l}(s, q)}{l!m!(k-l)!(n-k-m)!}(\alpha_j^m(p, s)\alpha_i^l(s, q) - \alpha_i^l(p, s)\alpha_j^m(s, q)),$$

Consider the matrix,

$$D_i^n := \begin{pmatrix} \alpha_i^n(1) & r_i^n(1,2) & \ldots & r_i^n(1,r) \\ r_i^n(2,1) & \alpha_i^n(2) & \ldots & r_i^n(2,r) \\ . & . & \ldots & . \\ r_i^n(r,1) & r_i^n(r,2) & \ldots & \alpha_i^n(r) \end{pmatrix}$$

with,

$$r_i^0(p,q) = 0, \; r_i^n(p,q) = \sum_{l=0}^{n} \binom{n}{l} \alpha_i^l(p,q)\sigma_{n-l}(p,q),$$

Then $\rho(d^n t_i) = D_i^n$ define a representation,

$$\rho : Ph^{\infty}(k[\underline{t}])) \to M_r(k)$$

Let us, as above, consider the matrix,

$$X_i = \rho(exp(\tau\delta))(t_i) = \sum_{n\geq 0} \tau^n/n! D_i^n$$

Putting

$$\alpha_i(p) = \sum_{n=0}^{\infty} \tau^n/n!\ \alpha_i^n(p),\ \ \alpha_i(p,q) = \sum_{n=0}^{\infty} \tau^n/n!\ \alpha_i^n(p,q),$$

and, $\sigma(p,q) = \sum_{n=0}^{\infty} \tau^n/n!\ \sigma^n(p,q)$, we find the explicite formulas,

$$X_i = \begin{pmatrix} \alpha_i(1) & \sigma(1,2)\alpha_i(1,2) & ... & \sigma(1,r)\alpha_i(1,r) \\ \sigma(2,1)\alpha_i(2,1) & \alpha_i(2) & ... & \sigma(2,r)\alpha_i(2,r) \\ ... & ... & ... & ... \\ \sigma(r,1)\alpha_i(r,1) & \sigma(r,2)\alpha_i(r,2) & ... & \alpha_i(r) \end{pmatrix},\ \ i = 1, ..., d.$$

Now, compute, and put,

$$[X_i, X_j] = 0,$$

We may consider the *space*

$$\mathbf{A}(r) = k[\alpha_i^n(p), \sigma_n(p,q)]/\mathfrak{a},$$

with coordinates $\{\alpha_i^n(p), \sigma_n(p,q), i = 1, ..., d, n \geqslant 0, p, q = 1, ..., r\}$, and where the ideal $\mathfrak{a}$ is generated by the equations above, as the versal base space for the versal family of the non-commutative deformation theory applied to the family of $Ph^\infty(k[\underline{t}]))$ modules defined by the object $\mathcal{P}$. Since $Ph^\infty(k[\underline{t}]))$ is infinitely generated, there is, strictly speaking, no such thing, but we shall see that in special cases, we can overcome this difficulty. In dimension $d = 3$ and order 1 the condition above reads:

$$\sigma_1(p,q)(\alpha^1(p,q)\times\alpha^0(p,q)) = -\sigma_1(p,s)\sigma_1(s,q)(\alpha^0(p,s)\times\alpha^0(s,q)), \forall p, s, q$$

This says that for any two of the three points in space, the relative momentum must sit in the plane defined by the three points, the length being determined by the 3 coupling constants. Moreover, the sum of all three relative momenta must be 0.

II The generic dynamical structure induced by a metric

II No 1

Let

$$C := k[t_1, ..., t_n]$$

Then,

$$Ph(C) = k < t_1, ..., t_n, dt_1, ..., dt_n > /([t_i, t_j], [dt_i, t_j] + [t_i, dt_j]).$$

A non-degenerate metric, $g = 1/2 \sum_{i=1}^{d} g_{i,j} dt_i dt_j \in Ph(C)$ induces an isomorphism of C-modules

$$\Theta_C = Hom_C(\Omega_C, C) \simeq \Omega_C.$$

Consider the bilateral ideal (σ_g) of $Ph(C)$ generated by

$$(\sigma_g) = ([dt_i, t_j] - g^{i,j}),$$

and put,

$$C(\sigma_g) := Ph(C)/(\sigma_g).$$

Let, moreover,

$$T := \sum_j T_j = -1/2 \sum_{i,j,l} \delta_{t_l}(g_{i,j}) g^{l,i} dt_j \tag{1}$$

$$= -1/2 (\sum_{k,l} \Gamma^k_{k,l} dt_l + \sum_{k,p,q} g^{k,q} \Gamma^p_{k,q} g_{p,l} dt_l), \tag{2}$$

and consider the inner derivation of $C(\sigma_g)$, defined by,

$$\delta := ad(g - T).$$

After a dull computation, we obtain, in $C(\sigma_g)$,

$$\delta(t_i) = dt_i, \ i = 1, ..., d.$$

Therefore, by universality, we have a well-defined dynamical structure, i.e. a δ stable ideal $(\sigma_g) \subset Ph^\infty(C)$, with Dirac derivation, $\delta = ad(g - T)$. It is also easy to see that (σ_g) is invariant w.r.t. isometries.

Relation to the spectral triples of Connes

II No 3

A spectral triple, $(\mathfrak{A}, \mathfrak{H}, \mathfrak{D})$, is formally, a (unitary)representation, of an associative complex algebra, in a complex Hilbert space,

$$\rho : \mathfrak{A} \to End_{\mathbf{C}}(\mathfrak{H})$$

together with an operator, called the Dirac operator, $\mathfrak{D} \in End_{\mathbf{C}}(\mathfrak{H})$. This induces a derivation,

$$ad(\mathfrak{D}) : \mathfrak{A} \to End_{\mathbf{C}}(\mathfrak{H}),\ ad(\mathfrak{D})(a) = [\mathfrak{D}, \rho(a)],$$

corresponding to a vanishing element of $Ext^1_{\mathfrak{A}}(\mathfrak{H}, \mathfrak{H})$, such that the extension of ρ with itself, defined by $ad(\mathfrak{D})$ is trivial. Clearly, it follows from this, that there is an extension of ρ from $\mathfrak{A}$ to $Ph(\mathfrak{A})$, i.e. a representation,

$$\rho_1 : Ph(\mathfrak{A}) \to End_{\mathbf{C}}(\mathfrak{H}),\ \rho_1(da) = [\mathfrak{D}, \rho(a)],$$

Therefore, given a metric g on the affine space defined by $\mathfrak{A}$ (in the commutative case) the Dirac operator must be given by the Dirac

II No 4 There are in $Ph(C)$ force laws, one is given by,

$$(*)\ d^2 t_i = -\sum_{p,q} \Gamma^i_{p,q} dt_p dt_q - 1/2 \sum_{p,q} g_{p,q}(F_{i,p} dt_q + dt_p F_{i,q})$$
$$+ 1/2 \sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)] dt_l + [dt_i, T]$$
$$= -\sum_{p,q} \Gamma^i_{p,q} dt_p dt_q - \sum_{p,q} g_{p,q} F_{i,p} dt_q + 1/2 \sum_{l,p,q} g_{p,q}[F_{i,q}, dt_p]$$
$$+ 1/2 \sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)] dt_l + [dt_i, T]$$

and consistent with the Dirac derivation in $C(\sigma_g)$, as well as with the classical equation for geodesics in $C(\sigma_0)$, where we have $\sigma_0 =< \{[dt_i, t_j], [dt_i, dt_j]\} >$, so that $C(\sigma_0) = Ph(C)_{com}$, the commutative classical phase space.

Consequences of the Force Laws, General Relativity

II No 5

We have got a Field Theory for connections on C-representations V, i.e. for representations,

$$\rho : C(\sigma_g) \rightarrow End_k(V)$$

with Dirac derivation for representations, $[\delta] = 0$ and Hamiltonian $Q = \rho(g - T)$.

We have also a Model for General Relativity, for the scheme $Spec(Ph(C)_{com})$, i.e. for representations,

$$\rho : C(\sigma_0) \rightarrow End_{C(\sigma_0)}(C(\sigma_0)) \rightarrow k$$

with Dirac derivation,

$$\delta = \sum_{i,p,q} (-\Gamma^i_{p,q} v_p v_p \delta_{v_i} + v_i \delta_{t_i})$$

Here v_i are the "vertical" coordinates in the phase space of $C = k[t_1, .., t_n]$, i.e. the momenta. The Hamiltonian Q is, obviously, trivial.

Further consequences of the Force Laws

II No 6

For the Euclidean or the Minkowski metric, the Force Law, or the equation of motion, reduces to,

$$d^2 t_i = -\sum_{p,q} g_{p,q} F_{i,p} dt_q + 1/2 \sum_{l,p,q} g_{p,q} [F_{i,q}, dt_p].$$

Given a representation,

$$\rho : Ph(C) \rightarrow End_k(\Theta_C),$$

this implies the following simple evolution equation,

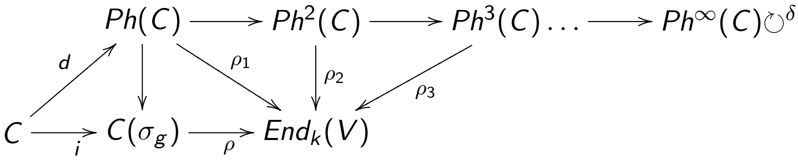
$$\dot{\rho}(dt_i) = \rho(d^2 t_i) = -\sum_{p} (F_{i,p} \rho(d_p) + 1/2 \nabla_{\delta_s}(F_{i,p})).$$

which, when we have introduced our favorite Toy Model, will give us the Maxwell equations.

Main Theorem

II No 7

Let **M** be the space (of isomorphism classes) of metrics on C. For every point $g \in \mathbf{M}$ consider the diagram,



Where ρ_1 is the representation of $Ph(C)$ induced by a representation ρ of $C(\sigma_g)$. Consider the family of k-algebras,

$$\mu : \mathbf{C}(\sigma) \to \mathbf{M}$$

indexed by the possible metrics of C, such that $g \in \mathbf{M}$ corresponds to $C(\sigma_g)$, and let:

$$\mathbf{T}_{\mathbf{M},g} = \{(h_{i,j})\}, \ h_{i,j} = h_{j,i} \in C.$$

be the tangent space to **M**, at g.

Define, $h^{i,j}$ by,

$$h_{i,j} = -\sum_{p,q} g_{i,p} h^{p,q} g_{q,j}, \ \ h = \{h_{i,j}\} \in \mathbf{T_M}.$$

Consider now the first order deformation $g + \epsilon h$, of the metric g, and the corresponding Dirac derivation, $ad(g + \epsilon h - T')$ in $C(\sigma_{(g+\epsilon h)})$. Then we find, in $C(\sigma_{(g+\epsilon h)})$,

$$[dt'_i, t_j] = g^{i,j} - h^{i,j}\epsilon = (g + \epsilon h)^{i,j}.$$

Moreover, the derivation $\eta : Ph(C) \to End_k(V)$ defined by,

$$\eta(t_i) = 0, \ \ \eta(dt_i) = \sum_{l,q} h^{i,l} \rho_1(g_{l,q} dt_q) = \sum_l h^{i,l} \nabla_{\delta_l}$$

induces an element,

$$\eta(h) \in Ext^1_{Ph(C)}(V, V)$$

Main Theorem

II No 9

This η produces an injective map,

$$\mathbf{T}_{\mathbf{M},g} \rightarrow Ext^1_{Ph(C)}(V, V).$$

onto the first order extensions of ρ_1.
In particular, any non-trivial deformation of the metric g induces a non-trivial deformation of the $Ph(C)$-representation (ρ_1, V), and any *first order* non-trivial deformation of the $Ph(C)$-representation (ρ_1, V) induces a non-trivial deformation of the metric.

Let $\mathfrak{g} \subset Der_k(A)$, be a sub Lie-module, and consider first, in the commutative case, the scheme (algebraic variety), $X = Spec(A)$, and the Lie algebra $\mathfrak{g}$ as a Lie algebra of vector fields defined on X. The set of maximal integral subschemes,

$$X/\mathfrak{g} := \{M \subset X | \Theta_M = \mathfrak{g}|M\},$$

is called the quotient of X by $\mathfrak{g}$, and coincides, in good cases with the quotient of X by the group of automorphisms $\mathbf{G}$ acting on X, with $\mathfrak{lie}\mathbf{G} = \mathfrak{g}$, when this exists. In the classical, commutative case, one would identify,

$$X/\mathfrak{g} = Spec(A^{\mathfrak{g}}),$$

where, $A^{\mathfrak{g}} := \{a \in A | \forall \gamma \in \mathfrak{g},\ \gamma(a) = 0\}$. This is, however, only reasonable when $\mathfrak{g}$ is reductive and/or all orbits of $\mathbf{G}$ are closed, which they rarely are.

Quotients in Non-commutative Algebraic Geometry

III No 2

In the non-commutative situation, this last definition of a quotient, has no meaning. The algebra A may have no 1-dimensional representations at all. The solution is to define the relevant points $Simp(A)$, of the geometry $\mathbf{A}$, defined by A, and then to seek out those points that should correspond to the points of the quotient $\mathbf{A}/\mathfrak{g}$.

Consider a representation $\rho : A \to End_k(V)$, and let for every $\gamma \in \mathfrak{g}$ the derivation, $\gamma \circ \rho \in Der_k(A, Hom_k(V, V))$, map to 0 in $Ext^1_A(V, V)$. This means that the representation ρ, is not moved by the action of $\mathfrak{g}$. As above, it follows from, $\kappa(\gamma \circ \rho) = 0$, that there exist an element, $Q_\gamma \in Hom_k(V, V)$, the Hamiltonian, such that for all $\gamma \in \mathfrak{g}$ and all $a \in A$,

$$\rho(\delta(a)) = Q_\gamma \circ \tilde{\rho}(a) - \rho(a) \circ Q_\gamma.$$

which can be written as,

$$Q_\gamma(av) = \gamma(a)v + aQ_\gamma(v), \forall v \in V,$$

i.e. Q is a $\mathfrak{g}$-connection on V.

Now, we define,

$$\mathbf{A}/\mathfrak{g} := Simp(\{\rho | \delta := \gamma \circ \rho = 0 \in Ext^1_A(V, V)\})$$

where *Simp* of course picks out those representations of this sort, with no such sub-representations.
The aim of, my kind, of non-commutative algebraic geometry is to to associate to any family of representations **V**, of the algebra A, the optimal extension-algebra,

$$\eta : A \to \mathbf{O}(\mathbf{V})$$

for which the family **V** becomes the set of simple $\mathbf{O}(\mathbf{V})$-representations.
Then **V** should be called: A Scheme for $\mathbf{O}(\mathbf{V})$.

Quotients in Geometry and Physics: Gauge Groups

III No 4

The notion of Gauge Group in physics, is intimately related to this non-commutative quotient structure.
We shall, in fact, show that there is an algebra H, representing our Cosmos, a Lie algebra, $\mathfrak{G}$ acting on $H(\sigma) := Ph(H)/([h, dh]|h \in H)$, the partially commutativization of $Ph(H)$, such that the main ingredients of the Standard Model, including representations like the Weyl and the Dirac Spinors pop up as points of the quotient,

$$H(\sigma)/\mathfrak{G},$$

suggesting that the standard Quantum Theory should be thought of as part of non-commutative algebraic geometry.

Global Gauge Groups

III No 5

Suppose, we have identified a k-Lie algebra $\mathfrak{g}_0 \subset Der_k(A)$, of infinitesimal automorphisms, i.e. of derivations of A, a *global gauge groupe*, leaving invariant the physical properties of our phenomena $\mathbf{P}$. We would then be led to consider the *quotient space* $\mathbf{M}/\mathfrak{g}_0$, which in our non-commutative geometry, is equivalent to restricting our representations, $\rho : A \to End_k(V)$, to those representations V for which, $\mathfrak{g}_0 \subset \mathfrak{g}_V$. This would then imply that the corresponding *Hamiltonians*, Q_γ define a $\mathfrak{g}_0$-connection on V,

$$Q : \mathfrak{g}_0 \longrightarrow End_k(V),$$

such that, for all $c \in A$, and for all $\gamma \in \mathfrak{g}_0$, $\rho(\gamma(c)) = [Q_\gamma, \rho(c)]$.

This is usually written,

$$\rho(\gamma(c)) = [\gamma, \rho(c)].$$

The curvature,

$$R(\gamma_1, \gamma_2) := [Q_{\gamma_1}, Q_{\gamma_2}] - Q_{[\gamma_1, \gamma_2]} \in End_A(V),$$

corresponds to a *global force* acting on the representation ρ. These forces, *mediated* by the *gauge-particles*, $\lambda \in \mathfrak{g}_0$, will be the first to be studied in some details. Put,

$$Rep(A, \mathfrak{g}_0) := \{\rho \in Rep(A)|,\ \kappa(\gamma\rho) = 0, \forall \gamma \in \mathfrak{g}_0\} = \{\rho \in Rep(A)|\mathfrak{g}_0 \subset \mathfrak{g}_\rho\},$$

where $Rep(A)$ is the category of all representations of C, and notice that, in the commutative situation, if we consider the case where the gauge group $\mathfrak{g}_0 = Der_k(A)$ then $Rep(A, \mathfrak{g}_0)$ is the category of *A-Connections*, for which the space of isomorphism classes is discrete. Notice that this is the situation in the classical quantum theory, where the *Hilbert Space* is always considered as the unique state space of interest.

Conclusions and Apology

The physical interpretation of the mathematical models presented here, and in the literature referred to in the next slide, should not, at the moment, be considered as a serious proposition for a new physics. With respect to the Big Bang, for example, I think any existing model, must be approached with utmost care.
However, I think the idea of defining time as a metric on the moduli space of the iso-classes of the mathematical models used to specify/define the physical phenomenon of interest, may be of some interest. I therefore proposed to name the study of the geometry of reasonable moduli spaces of this kind:
Geometry of Time-Spaces.

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