Unimodular ICIS, A CLASSIFIER

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Algebraic Geometry and its Applications,

ASSMS G. C.U. Lahore

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Outline

Motivation

We present the algorithms for computing the normal form of unimodular complete intersection surface singularities classified by C. T. C. Wall. He indicated in the list only the μ-constant strata and not the complete classification in each case. We give a complete list of surface unimodular singularities. We also give the description of a classifier which is implemented in computer algebra system SINGULAR .

Outline

Definitions

Problem

Unimodular complete intersections

Algorithm

Example

Let $K[[x]] = K[[x_1, ..., x_n]]$ is the local ring of formal power series and $\mathfrak{m} = \langle x_1, ..., x_n \rangle$ is its maximal ideal over K an algebraically closed field.

$g \in K[[x]]$ is a hypersurface singularity if $g \in \mathfrak{m}^2$ and $g \neq 0$

The ideal

$$j(g) := \langle g_{x_1}, \ldots g_{x_n} \rangle$$

is called Jacobian ideal or the Milnor ideal of g and

$$tj(g) := \langle g \rangle + j(g)$$

is called the Tjurina ideal of g. The $K - algebras$

$$M_g = K[[x]]/j(g), \quad T_g = K[[x]]/tj(f)$$

is the Milnor and Tjurina Algebra of g, respectively. The numbers

$$\mu(g) = dim_K(M_g), \quad \tau(g) = dim_K(T_g)$$

are called the Milnor and Tjurina numbers of g, respectively.

Let $f_1, f_2 \in K[[x]]$.

f_1 is called right equivalent to f_2, $f_1 \sim_r f_2$ if $\exists$ an automorphism ϕ of $K[[x]]$ s.t $f_2 = \phi(f_1)$.

f_1 is called contact equivalent to f_2, $f_1 \sim_c f_2$ if $\exists$ an automorphism ϕ of $K[[x]]$ and u a unit $u \in K[[x]]^*$ s.t $f_2 = u.\phi(f_1)$.

Blowing up is defined in coordinates as follows

$$Bl_0\mathbb{C}^n = \{(x_1, \ldots, x_n), (u_1, \ldots, u_n) | x_i u_j = x_j u_i \quad \forall i, j\} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$$

$(u_1, \ldots, u_n)$ are the homogenous coordinates of $\mathbb{P}^{n-1}$. Then there is a projection map

$$\pi : Bl_0\mathbb{C}^n \to \mathbb{C}^n$$

$$\pi : (x_1, \ldots, x_n, u_1, \ldots, u_n) \to (x_1, \ldots, x_n)$$

which is called the *blowing up map*.
$E := \pi^{-1}(0)$ the *exceptional divisor* of π.

Let $f =< f_1, f_2, ..., f_p >$. It is called complete intersection if $\dim K[[x]]/\langle f_1, \ldots, f_i \rangle = n - i \ , \forall \ \ i$.

Let $I_{n,p}$ be the set of all isolated complete intersection singularities. If $f = (f_1, \ldots, f_p), g = (g_1, \ldots, g_p) \in I_{n,p}$, $G_c = Aut(K[[x]]) \times GL_p(K[[x]])$.
Then G_c acts on $I_{n,p}$ as follows:
let $(\phi, \psi) \in G_c$ such that $(\phi, \psi)(f) = \psi^{-1} \circ g \circ \phi$.

Let f and $g \in I_{n,p}$ are called contact equivalent, if there exists $(\phi, \psi) \in G_c, f = (\phi, \psi)(g)$.

Let $f = \langle f_1, \dots, f_p \rangle \subseteq K[[x]]$ be a complete intersection. $f = \langle f_1, \dots, f_p \rangle$ has an isolated singularity at 0, if

1. $\langle f_1, \dots, f_p, M_1, \dots, M_k \rangle \subseteq \mathfrak{m}$, $M_1, \dots, M_k$ the $p \times p$–minors of $(\frac{\partial f_i}{\partial x_j})$.
2. $\mathfrak{m}^c \subseteq \langle f_1, \dots, f_p, M_1, \dots, M_k \rangle$ for some $c > 0$.

The *Milnor number* $\mu(f)$ is defined as follows

$$\mu(f) = \sum_{i=1}^{p} (-1)^{p-i} dim_K K[[x]]/C_i$$

with $C_i = \langle f_1, f_2, \dots, f_{i-1}, \frac{\partial(f_1, \dots, f_i)}{\partial x_{j_1} \dots x_{j_i}}, 1 \leq j_1 \dots < j_i \leq n \rangle$.
The *Tjurina number* of f is defined to be

$$dim_K K[[x]]^p / f K[[x]]^p + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} K[[x]].$$

An element $f \in I_{n,p}$ is called simple singularity, if there exists a neighborhood of f in $I_{n,p}$ containing only finitely many orbits of G_c. In other words the modality of the singularity is zero.

f is called unimodular singularity if there exists a neighborhood of f in $I_{n,p}$ containing only one–dimensional families of orbits of G_c.

Let $f = \langle f_1, \ldots, f_p \rangle \in I_{n,p}$. Then $F = \langle F_1, \ldots, F_p \rangle, \quad F_i \in K[[x,t]]$ where $t = \{t_1, \ldots, t_n\}$ is a deformation of f if

$$K[[x]]/\langle f \rangle \cong K[[x]]/\langle F(x,0) \rangle.$$

Any deformation can be induced from the versal deformation by specifying parameters.
$F = f + \sum t_i m_i$ is a *versal deformation of f* where $m_1, \ldots, m_\tau$ is basis for

$$K[[x]]^p / fK[[x]]^p + \begin{pmatrix} \partial f_1/\partial x_1 \\ \vdots \\ \partial f_p/\partial x_1 \end{pmatrix} K[[x]] + \ldots + \begin{pmatrix} \partial f_1/\partial x_n \\ \vdots \\ \partial f_p/\partial x_n \end{pmatrix} K[[x]].$$

Outline

Main Research Problem

- Complete the list of unimodular surface singularities given by C.T.C. Wall and characterize the singularities in terms of certain invariants.
- Develop software packages for computing the type of singularities

Related Work

- Classification of singularises with respect to right and contact equivalence
  - ADE singularities, classification of unimodular singularities in characteristic zero,. . .
    Arnold.
  - Hypersurface simple singularities in positive characteristic with respect to right and contact equivalence,
    Gert-Martin Greuel, Kröning and Nguyen Hong Duc.
  - Classification of simple isolated complete intersection singularities for characteristic zero
    Marc Giusti.
  - Classification of unimodal isolated singularities complete intersections for characteristic zero,
    C. T. C. Wall.

Related Work

- Singular package
  - Classifier for the hypersurface singularities given by Arnold
    Kai Krueger.
    https://github.com/Singular/Sources/blob/spielwiese/Singular/LIB/classify.lib
  - Unimodular isolated complete intersection curve singularities
  - D. Afzal and G. Pfister. A classifier for unimodular complete intersection space curve singularities.
    https://content.sciendo.com/view/journals/auom/24/1/article-p95.xml

  - D. Afzal and G. Pfister. *classifyci1.lib*. A SINGULAR 3-1-6 library for classifying unimodular isolated complete singularities for the base field of characteristic 0.

Characterization of normal form of 2-jet of singularities

Consider $I = \langle f_1, f_2 \rangle \subseteq \langle x, y, z, w \rangle^2 \mathbb{C}[[x, y, z, w]]$ defining a complete intersection singularity and let I_2 be the 2-jet of I.

Let $\bigcap_{i=1}^{s} Q_i$ be the irredundant primary decomposition of I_2 in $\mathbb{C}[[x, y, z, w]]$. Let t be the number of prime ideals appearing in primary decomposition of I_2 and j_i be the number of conjugates corresponding to each prime ideal. Let $d_i = dim_{\mathbb{C}} \mathbb{C}[[x, y, z, w]]/Q_i, i = 1, ..., s$ and h_i be the Hilbert polynomial of $\mathbb{C}[[x, y, z, w]]/Q_i$.

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Name	*Characterization*	*Normalform*
I	$s = 4, d_1 = d_2 = d_3 = d_4 = 1$ $h_1 = h_2 = h_3 = h_4 = 1 + t$	$\langle xy - xz, yz - yx \rangle$
$J^{'}$	$s = 1$ $d_1 = 2, h_1 = 4t$ $t = 1, j_1 = 1$ A_2 *after blowing up*	$\langle xy + z^2, w^2 + xz \rangle$

Table: part of charecterization table of normal forms of $2 - jet$ of I

[0]Here after blowing up means the singularity type of the strict transform of I_2 in the blowing up of $\langle x, y, z, w \rangle$.

Outline

We set

$$l_i(x,y) = \begin{cases} xy^q, & \text{if } i = 2q \\ y^{q+2}, & \text{if } i = 2q+1 \end{cases}$$

for brevity. Assume the $2-jet$ of $< \langle f,g \rangle$ has normal form $\langle xy - xz, yz - xy \rangle$. In this case according to C.T.C.Wall's classification the unimodular surface singularities with their Milnor number say μ and Tjurina number τ are given in the table below.

Name	*Normal form*	μ	τ
$I_{1,0}$	$\langle x(y-z)+w^3, y(z-x)+\lambda w^3 \rangle \quad \lambda \neq 0,1$	13	13
$I_{1,i}$	$\langle x(y-z)+w^3, y(z-x)+w^2 l_{i-1}(x,w) \rangle$	$13+i$	$13+i-2$

Table:

Proposition

The unimodular complete intersection surface singularity with Milnor number $\mu = 13$ are $I_{1,0}$ with Tjurina number $\tau = 13$ defined by the ideal

$$\langle xy - xz + w^3, yz - xy + \lambda w^3 \rangle$$

and $I_{1,0,1}$ with Tjurina number $\tau = 12$ defined by the ideal

$$\langle xy - xz + w^3, yz - xy + \lambda w^3 + w^4 \rangle .$$

Summarizing the results of the above preposition we complete the list of unimodular complete intersection singularities in case of $\langle f, g\rangle$ having 2-jet with normal forms $\langle xy - xz, yz - xy\rangle$.

Type	*Normalform*	μ	τ
$I_{1,0,1}$	$\left\langle xy - xz + w^3, yz - xy + \lambda w^3 + w^4\right\rangle$	13	12

Table:

PROPOSITION

Let $(V(\langle f, g\rangle), 0) \subseteq (\mathbb{C}^4, 0)$ be the germ of a complete intersection surface singularity. Assume it is not a hypersurface singularity and the 2-jet of $\langle f, g\rangle$ has normal form $\langle xy - xz, yz - xy\rangle$. $(V(\langle f, g\rangle), 0)$ is unimodular if and only if it is isomorphic to a complete intersection in Tables above.

Outline

Algorithm 1 `Isingularity(I)`

Input: $I = < f, g > \subseteq < x, y, z, w >^2 \mathbb{C}[[x, y, z, w]]$ such that 2-jet of I has normal form $< xy - xz, yz - xy >$

Output: the type of the singularity

```
compute μ =Milnor number of I;
compute τ =Tjurina number of I;
compute B =the singularity type of the strict transform of I in the blowing up
    of ⟨x, y, z, w⟩ [1]
if μ = 13 and B = A[1] then
    if μ − τ = 0 then
        return (I_{1,0});
    if μ − τ = 1 then
        return (I_{1,0,1});
if μ = 13 + i, i > 0 and B = A[i + 1] then
    if μ − τ = 2 then
        return (I_{1,i});
return (not unimodular);
```

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Example

Example

$ring \quad R = 0, (x, y, z, w), ds;$
$ideal \quad f = \langle x(y-z) + w^3, y(z-x) + \lambda w^3 \rangle;$
$map \quad \phi = R, x+y+z, y-2z, z, w;$
$ideal \quad f = \phi(f);$

We want to classify this complete intersection singularity.

Step 1:

Compute Milnor number.

$\mu(f) = 13$

Step 2:

Compute $jet^2(f)$.

Step 3:

Find the type of the ideal J.

From the characterization table of 2-jet of f

Step 4:
Compute Tjurina number of f.
$\tau(f) = 13$
Step 5:
compute $B =$the singularity type of the strict transform of I in the blowing up of $\langle x, y, z, w\rangle$
$B = A[1]$
$\Rightarrow$ f is unimodular of Type $I_{1,0}$.

Thank you!