Lecture II: Geodesics and Covariant Derivatives

We consider a parametrized surface  $\mathbf{r}(u, v)$  given by two parameters u and v.

For example,

$$\mathbf{r}(\theta,\phi) = R\left(\sin(\theta)\cos(\phi),\sin(\theta)\sin(\phi),\cos(\theta)\right)$$
(1)



Figure 1: Tangent vectors in the coordinate directions spanning the tangent plane.

The tangent vectors at the point labelled by (u, v) are given by

$$\mathbf{r}_{u} = \frac{\partial \mathbf{r}}{\partial u} \qquad \qquad \mathbf{r}_{v} = \frac{\partial \mathbf{r}}{\partial v} \qquad (2)$$

The metric on the surface is given by considering the the infinitesimal

vector:

$$d\mathbf{r} = \mathbf{r}(u + du, v + dv) - \mathbf{r}(u, v)$$
(3)

$$= \mathbf{r}_{u}du + \mathbf{r}_{v}dv + \frac{1}{2}\mathbf{r}_{uu}du^{2} + \mathbf{r}_{uv}dudv + \frac{1}{2}\mathbf{r}_{vv}dv^{2} + \cdots$$
(4)

$$ds^{2} = d\mathbf{r} \cdot d\mathbf{r} = g_{11}du^{2} + (g_{12} + g_{21})dudv + g_{22}dv^{2} \qquad (5)$$
$$g_{ab} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{r}_{u} \cdot \mathbf{r}_{u} & \mathbf{r}_{v} \cdot \mathbf{r}_{v} \\ \mathbf{r}_{v} \cdot \mathbf{r}_{u} & \mathbf{r}_{v} \cdot \mathbf{r}_{v} \end{pmatrix}$$

**Example:** For a sphere the tangent vector in the direction of increasing  $\theta$  and  $\phi$  are given by:

$$\mathbf{r}_{\theta} = \frac{\partial \mathbf{r}}{\partial \theta} = R\Big(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta)\Big)$$
(6)  
$$\mathbf{r}_{\phi} = \frac{\partial \mathbf{r}}{\partial \phi} = R\Big(-\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), 0\Big)$$

$$g_{ab} = \begin{pmatrix} R^2 & 0\\ 0 & R^2 \sin^2(\theta) \end{pmatrix}$$
(7)

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2(\theta) d\phi^2 \tag{8}$$

Given any curve on the surface we can determine the length of the curve using the metric. If the curve is given (u(t), v(t)) then

$$\mathcal{L}(t_1, t_2) = \int_{t_1}^{t_2} f(t) dt \tag{9}$$
$$f(t) = \sqrt{g_{ab}} \frac{du^a}{dt} \frac{du^b}{dt}$$

Let  $\gamma(t)$  be a curve on the surface parameterized by some parameter t. If we denote the surface by S then:



Figure 2: A curve on the surface S and the tangent vector to the curve at a point.

Then the tangent vector to the curve is given by the derivative of  $\gamma(t)$  with respect to t (as we saw last time):

$$\gamma'(t) = \mathbf{r}_u \frac{du}{dt} + \mathbf{r}_v \frac{dv}{dt}$$

$$= u'(t) \mathbf{r}_u + v'(t) \mathbf{r}_v$$
(11)

$$\mathbf{t} = \gamma'(t) \qquad \mathbf{r}_u = \mathbf{r}_1, \, \mathbf{r}_v = \mathbf{r}_2 \tag{12}$$

$$\mathbf{t} = \frac{du^a}{dt} \,\mathbf{r}_a \tag{13}$$

The tangent vector lives entirely in the tangent plane spanned by  $\mathbf{r}_1$  and  $\mathbf{r}_2$ . Now lets consider how this tangent vector changes as

we move along the curve. We consider the derivative of the tangent vector (or the acceleration along the curve):

$$\gamma''(t) = \mathbf{t}' \tag{14}$$

$$\gamma''(t) = \frac{d^2 u^a}{dt^2} \mathbf{r}_a + \frac{du^a}{dt} \left( \mathbf{r}_{ab} \frac{du^b}{dt} \right)$$
(15)  
$$= \frac{d^2 u^a}{dt^2} \mathbf{r}_a + \frac{du^a}{dt} \frac{du^b}{dt} \mathbf{r}_{ab}$$

The first part is a vector which lies in the tangent plane but the second part may not entirely lie in the tangent plane.

The vector  $\mathbf{r}_{ab}$  is the derivative of the tangent vector  $\mathbf{r}_a$  with respect to  $u^b$ . Since  $\mathbf{r}_1, \mathbf{r}_2$  together with the unit normal vector  $\mathbf{n}$  ( $\sim \mathbf{r}_1 \times \mathbf{r}_2$ ) form a basis therefore:

$$\mathbf{r}_{ab} = \Gamma^c_{ab} \mathbf{r}_c + E_{ab} \,\mathbf{n} \tag{16}$$

The coefficients  $\Gamma_{ab}^c$  are called the Christoffel symbols,

g

$$\mathbf{r}_{d} \cdot \mathbf{r}_{ab} = \Gamma^{c}_{ab} g_{dc}$$
(17)  
$${}^{de}\mathbf{r}_{d} \cdot \mathbf{r}_{ab} = \Gamma^{c}_{ab} \delta^{e}_{c} = \Gamma^{e}_{ab}$$

We can obtain an expression for the Christoffel symbols entirely in terms of the metric. Recall that

$$g_{ab} = \mathbf{r}_a \cdot \mathbf{r}_b \tag{18}$$

differentiating this with respect to  $u^c$  we get  $(g_{ab,c} = \frac{\partial g_{ab}}{\partial u^c})$ 

$$g_{ab,c} = \mathbf{r}_{ac} \cdot \mathbf{r}_b + \mathbf{r}_a \cdot \mathbf{r}_{bc} \tag{19}$$

similarly

$$g_{ac,b} = \mathbf{r}_{ab} \cdot \mathbf{r}_c + \mathbf{r}_a \cdot \mathbf{r}_{cb}$$

$$g_{cb,a} = \mathbf{r}_{ca} \cdot \mathbf{r}_b + \mathbf{r}_c \cdot \mathbf{r}_{ba}$$
(20)

then

$$egin{aligned} g_{ac,b}+g_{cb,a}-g_{ab,c}&=&\mathbf{r}_{ab}\cdot\mathbf{r}_{c}+\mathbf{r}_{a}\cdot\mathbf{r}_{cb}+\mathbf{r}_{ca}\cdot\mathbf{r}_{b}+\mathbf{r}_{c}\cdot\mathbf{r}_{ba}\ &-(\mathbf{r}_{ac}\cdot\mathbf{r}_{b}+\mathbf{r}_{a}\cdot\mathbf{r}_{bc})\ &=&2\mathbf{r}_{c}\cdot\mathbf{r}_{ab}=2\Gamma^{d}_{ab}g_{dc}\ &\Gamma^{d}_{ab}&=&rac{1}{2}g^{dc}\Big(g_{ac,b}+g_{cb,a}-g_{ab,c}\Big) \end{aligned}$$

# Example

$$\mathbf{r}(\theta,\phi) = R\Big(\sin(\theta)\cos(\phi),\sin(\theta)\sin(\phi),\cos(\theta)\Big)$$
(21)

$$\mathbf{r}_{\theta} = \frac{\partial \mathbf{r}}{\partial \theta} = R\Big(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta)\Big) \quad (22)$$
$$\mathbf{r}_{\phi} = \frac{\partial \mathbf{r}}{\partial \phi} = R\Big(-\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), 0\Big)$$

$$\mathbf{r}_{\theta\theta} = \frac{\partial \mathbf{r}_{\theta}}{\partial \theta} = \frac{\partial \mathbf{r}}{\partial \theta} = R\left(-\sin(\theta)\cos(\phi), -\sin(\theta)\sin(\phi), -\cos(\theta)\right)$$
$$= -\mathbf{r}$$
$$\mathbf{r}_{\phi\phi} = -R\left(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), 0\right)$$
$$\mathbf{n} \cdot \mathbf{r}_{\phi\phi} = -R\sin^{2}(\theta)$$
$$\mathbf{r}_{\theta\phi} = \mathbf{r}_{\phi\theta} = R\left(-\cos(\theta)\sin(\phi), \cos(\theta)\cos(\phi), 0\right)$$
$$\mathbf{n} \cdot \mathbf{r}_{\theta\phi} = 0$$

$$\Gamma^{\theta}_{\theta\theta} = \Gamma^{\phi}_{\theta\theta} = 0 \tag{23}$$

$$\Gamma^{\theta}_{\theta\phi} = g^{d\theta} \mathbf{r}_{d} \cdot \mathbf{r}_{\theta\phi} = g^{\theta\theta} \mathbf{r}_{\theta} \cdot \mathbf{r}_{\theta\phi} + g^{\phi\theta} \mathbf{r}_{\phi} \cdot \mathbf{r}_{\theta\phi}$$
(24)  
=  $R^{-2} \times 0 + 0 \times R^{2} \sin(\theta) \cos(\theta) = 0$ 

$$\Gamma^{\theta}_{\phi\phi} = g^{d\theta} \mathbf{r}_{d} \cdot \mathbf{r}_{\phi\phi} = g^{\theta\theta} \mathbf{r}_{\theta} \cdot \mathbf{r}_{\phi\phi} + g^{\phi\theta} \mathbf{r}_{\phi} \cdot \mathbf{r}_{\phi\phi}$$
(25)  
$$= R^{-2} \times \left( -R^{2} \sin(\theta) \cos(\theta) \right) = -\sin(\theta) \cos(\theta)$$

similarly

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\theta\phi} = \cot(\theta) \tag{26}$$

Thus the derivative of any tangent vector along a curve can be written as:

$$\gamma''(t) = \frac{d^{2}u^{a}}{dt^{2}}\mathbf{r}_{a} + \frac{du^{a}}{dt}\frac{du^{b}}{dt}\mathbf{r}_{ab}$$
(27)  
$$= \frac{d^{2}u^{a}}{dt^{2}}\mathbf{r}_{a} + \frac{du^{a}}{dt}\frac{du^{b}}{dt}\left(\Gamma_{ab}^{c}\mathbf{r}_{c} + E_{ab}\mathbf{n}\right)$$
$$= \underbrace{\left(\frac{d^{2}u^{a}}{dt^{2}}\mathbf{r}_{a} + \frac{du^{a}}{dt}\frac{du^{b}}{dt}\Gamma_{ab}^{c}\mathbf{r}_{c}\right)}_{\text{tangent to surface}} + \underbrace{\frac{du^{a}}{dt}\frac{du^{b}}{dt}E_{ab}\mathbf{n}}_{\text{orthogonal to surface}}$$

The curve for which the tangent vector is constant with respect to the surface i.e., the derivative of the tangent vector has no component along the surface are called geodesics. They are the analogs of straight lines in the plane. The curves of minimal length between two points are also geodesics.

Recall that we defined the curvature of a space curve as the length of the derivative of the tangent vector. In the above case we define two different notions of curvature called the normal curvature and the geodesic curvature.

$$\kappa_{normal} = \gamma''(t) \cdot \mathbf{n} = \frac{du^a}{dt} \frac{du^b}{dt} E_{ab}$$
(28)

The geodesic curvature is the part coming from the tangential component of  $\gamma''(t)$ . Suppose that the curve is parametrized by the arc length. Then the acceleration is orthogonal to the tangent vector since the tangent vector is a unit vector. Thus the tangential component of the acceleration  $\gamma''(t)$  is orthogonal to  $\mathbf{n}$  and  $\mathbf{r}_a$ . Thus it is along  $\mathbf{n} \times \mathbf{t}$ . The geodesic curvature is then defined as the component of the tangential part of  $\gamma''(t)$  along  $\mathbf{n} \times \gamma'(t)$ :

$$\gamma''(t) = \kappa_{geodesic} \,\mathbf{n} \times \mathbf{t} + \kappa_{normal} \,\mathbf{n} \tag{29}$$

If we consider the curve  $\gamma(t)$  just as a space curve then its curvature  $\kappa$  is related to normal and geodesic curvature as:

$$\kappa^2 = \kappa_{geodesic}^2 + \kappa_{normal}^2 \tag{30}$$

The equation of the geodesic can be written as:

$$\frac{d^2 u^a}{dt^2} \mathbf{r}_a + \frac{du^a}{dt} \frac{du^b}{dt} \Gamma^c_{ab} \mathbf{r}_c = 0$$
(31)

$$\frac{d^2 u^a}{dt^2} \mathbf{r}_a + \frac{du^c}{dt} \frac{du^b}{dt} \Gamma^a_{cb} \mathbf{r}_a = 0$$
(32)

$$\left(\frac{d^2u^a}{dt^2} + \frac{du^c}{dt}\frac{du^b}{dt}\,\Gamma^a_{cb}\right)\mathbf{r}_a = 0 \tag{33}$$

$$\left(\frac{d^2u^a}{dt^2} + \frac{du^c}{dt}\frac{du^b}{dt}\Gamma^a_{cb}\right) = 0$$
(34)

We can define the covariant derivative along the some direction given by a vector  $(V^1, V^2)$  as the directional directive projected to the tangent plane and is defined as:

$$\nabla_a \mathbf{r}_b = \Gamma^c_{ab} \mathbf{r}_c \tag{35}$$

 $V = V^a \mathbf{r}_a$  then

$$\nabla_{V}W = V^{a}\nabla_{a}W = V^{a}\nabla_{a}(W^{b}\mathbf{r}_{b}) = V^{a}(W^{b}_{,a}\mathbf{r}_{b} + W^{b}\Gamma^{c}_{ab}\mathbf{r}_{c})$$
$$= V^{a}\Big(W^{b}_{,a} + W^{c}\Gamma^{b}_{ca})\mathbf{r}_{b}$$
(36)

$$\nabla_a W^b = W^b_{,a} + W^c \Gamma^b_{ca} \tag{37}$$

A geodesic is a curve for which the covariant derivative of the tangent vector in the direction of the tangent vector is zero:

$$\nabla_{\mathbf{t}} \mathbf{t} = 0 \tag{38}$$

The covariant derivative is just the ordinary derivative along a vector taken with respect to the manifold. Let C be a curve parametrized by t with coordinates  $x^a(t)$  and let A be a vector field defined in the neighbourhood of the curve C. Then the derivative of A along the curve C is by

$$\frac{dA}{dt} = \frac{dx^{a}}{dt} \frac{\partial A(x)}{\partial x^{a}}$$

$$= \frac{dx^{a}}{dt} \frac{\partial A^{b}e_{b}}{\partial x^{a}} = \frac{dx^{a}}{dt} \left( A^{b}_{,a}e_{b} + A^{b}e_{b,a} \right)$$

$$= \frac{dx^{a}}{dt} \left( A^{b}_{,a}e_{b} + A^{b}\Gamma^{c}_{ab}e_{c} \right)$$
(39)

where we have as defined before  $\Gamma^c_{ba}$  such that

$$\frac{\partial e_b}{\partial x^a} = \Gamma^c_{ab} e_c \,. \tag{40}$$

Since  $\frac{d x^a}{dt}$  are the components of the tangent vector to the curve let us denote them by  $t^a$ ,

$$\frac{dA}{dt} = t^a \nabla_a A,$$

$$\nabla_a A = A^b_{;a} e_b$$
(41)

 $t^a A^b_{;a}$  are the components of the covariant derivative of A in the direction of the tangent vector to the curve,

$$A^{b}_{;a} = A^{b}_{,a} + \Gamma^{b}_{ac} A^{c} \,. \tag{42}$$

If the vector field A is constant along the curve C then  $\frac{dA}{dt} = 0$  which implies

$$t^a \nabla_a A = 0 \iff t^a \left( A^b_{,a} + \Gamma^b_{ac} A^c \right) = 0.$$
 (43)

#### Example 1: Geodesics on a sphere:

Consider the sphere with coordinates  $u^1 = \theta$ ,  $u^2 = \varphi$ . The sphere is given by

$$\mathbf{r}(\theta,\varphi) = (\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta).$$
(44)

The metric on the sphere is given by

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = d\theta^2 + \sin^2\theta \, d\varphi^2 \tag{45}$$

The non-zero christoffel symbols  $\Gamma^f_{ab}$  which we calculated earlier are:

$$\Gamma_{22}^{1} = -\sin\theta\cos\theta \qquad (46)$$
  
$$\Gamma_{12}^{2} = \Gamma_{21}^{1} = \cot\theta$$

Using the above in Eq.(34) we get

$$\frac{d^2\theta}{ds^2} - \sin\theta\cos\theta\left(\frac{d\,\varphi}{ds}\right)^2 = 0\,, \quad \frac{d^2\varphi}{ds^2} + 2\cot\theta\,\frac{d\,\theta}{ds}\,\frac{d\,\varphi}{ds} = 0 \quad (47)$$

The second equation can be written as

$$\frac{d}{ds}\left(\sin^2\theta \,\frac{d\,\varphi}{ds}\right) = 0 \implies \frac{d\,\varphi}{ds} = \frac{A}{\sin^2\theta}\,. \tag{48}$$

To solve the first equation we write  $\theta$  as a function of  $\varphi$  (prime denotes differentiation with respect to  $\varphi$ ):

$$\frac{d\theta}{ds} = \theta' \frac{d\varphi}{ds}$$

$$\frac{d^2\theta}{ds^2} = \theta' \frac{d^2\varphi}{ds^2} + \theta'' \left(\frac{d\varphi}{ds}\right)^2$$
(49)

Then the first equation becomes

$$\left(\theta'' - (\theta')^2 2\cot\theta - \sin\theta\cos\theta\right) \left(\frac{A}{\sin^2\theta}\right)^2 = 0 \tag{50}$$

One solution is A = 0 which implies  $\varphi = constant$  and  $\theta = \alpha s + \beta$ . These are the intersection of the sphere with the plane which contains the z-axis. Other solutions are given by

$$\theta'' - (\theta')^2 2 \cot \theta - \sin \theta \cos \theta = 0$$
(51)  

$$(\csc^2 \theta \theta')' = \cot \theta$$
  

$$(-\cot \theta)'' = \cot \theta$$
  

$$\cot \theta = B \cos(\varphi - \varphi_0)$$

We can rearrange the above equation to

$$\alpha \sin \theta \cos \varphi + \beta \sin \theta \sin \varphi + \gamma \cos \theta = 0.$$
 (52)

which in terms of x, y, z is give by

$$\alpha x + \beta y + \gamma z = 0, \qquad (53)$$

and represents a plane passing through the origin. Thus the geodesics on the sphere are the great circles (intersection of the sphere with the plane passing through the origin).

**Example 2: Geodesics on the hyperbolic plane:** The hyperbolic plane is the surface  $S = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$  with metric:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$
(54)

This is called the hyperbolic metric,

$$g_{ab} = \begin{pmatrix} y^{-2} & 0\\ 0 & y^{-2} \end{pmatrix}$$

The inverse metric is given by:

$$g^{ab} = \begin{pmatrix} y^2 & 0\\ 0 & y^2 \end{pmatrix} \tag{55}$$

The Christoffel symbols which are non-vanishing are given by:

$$\Gamma_{12}^{1} = \Gamma_{21}^{1} = \frac{1}{2}g^{11}(g_{11,2}) = -y^{-1}$$

$$\Gamma_{11}^{2} = \frac{1}{2}g^{22}(-g_{11,2}) = y^{-1}$$

$$\Gamma_{22}^{2} = \frac{1}{2}g^{22}(g_{22,2}) = -y^{-1}$$
(56)

The geodesic equations are then given by:

$$\frac{d^2x}{ds^2} - 2y^{-1}\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) = 0$$

$$\frac{d^2y}{ds^2} + y^{-1}\left(\frac{dx}{ds}\right)^2 - y^{-1}\left(\frac{dy}{ds}\right)^2 = 0$$
(57)

Using

$$\frac{dy}{ds} = \frac{dy}{dx}\frac{dx}{ds} \qquad \frac{d^2y}{ds^2} = \frac{dy}{dx}\frac{d^2x}{ds^2} + \frac{d^2y}{dx^2}\left(\frac{dx}{ds}\right)^2 \tag{58}$$

we get

$$\ddot{x} - 2y^{-1}(\dot{x})^2 y' = 0 \quad \text{where} \quad \dot{x} = \frac{dx}{ds} \text{ and } y' = \frac{dy}{dx}$$
$$y'\ddot{x} + y''(\dot{x})^2 + y^{-1}(\dot{x})^2 - y^{-1}(y')^2 \dot{x}^2 = 0$$

The second becomes:

$$(\dot{x})^{2} \left[ y'' + y^{-1} - y^{-1} (y')^{2} + 2y^{-1} (y')^{2} \right] = 0$$
  
$$\dot{x} = 0 \qquad \text{or} \qquad y \, y'' + 1 + (y')^{2} = 0 \tag{59}$$

The first equation represents the geodesics which are straight vertical lines. The second equation can be solved by taking y' = u and obtaining:

$$y \, u \, \frac{du}{dy} + 1 + u^2 = 0 \implies \frac{dy}{y} = -\frac{u du}{1 + u^2} \tag{60}$$
$$\ln(y) = -\frac{1}{2} \ln(1 + u^2) + c \implies y^{-2} = e^c (1 + u^2)$$

Define  $e^c = A^{-2}$  then

$$\frac{dy}{dx} = \pm \sqrt{A^2 y^{-2} - 1} \implies \frac{dy}{\sqrt{A^2 y^{-2} - 1}} = \pm dx$$
(61)  
$$\frac{y dy}{\sqrt{A^2 - y^2}} = \pm dx \implies -\sqrt{A^2 - y^2} = \pm x + B$$
  
$$(A^2 - y^2) = (x \mp B)^2 \implies (x - B)^2 + y^2 = A^2$$

Thus these geodesics are semi-circles with origin on the x-axis.



Geodesic equation from variational principle

Let U be an open set and  $C \subset U$  a curve parametrized by t. Let  $x^a$  be the coordinates on U so that in these coordinates the curve is given by  $x^a(t)$ . The arc-length of the curve between the point P and Q is given by

$$s = \int_0^1 \sqrt{g_{ab} \dot{x}^a \dot{x}^b} \, dt \,, \tag{62}$$

where t(P) = 0, t(Q) = 1 and dot denotes differentiation with respect to t. C is a geodesic if the first order variation of the arc length vanishes for it,

$$\delta s = \int_0^1 \left( \frac{\partial L}{\partial x^c} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^c} \right) \delta x^c \, dt = 0 \,, \tag{63}$$

where  $L = \sqrt{g_{ab} \dot{x}^a \dot{x}^b}$ . This gives the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^c} - \frac{\partial L}{\partial x^c} = 0.$$
(64)

Using the L given above we get

$$\frac{d}{dt} \left( \frac{1}{2\sqrt{g_{ab}\dot{x}^{a}\dot{x}^{b}}} g_{ab}\delta^{a}_{c}\dot{x}^{b} + g_{ab}\dot{x}^{a}\delta^{b}_{c} \right) - \frac{1}{2\sqrt{g_{ab}\dot{x}^{a}\dot{x}^{b}}} g_{ab,c}\dot{x}^{a}\dot{x}^{b} = 0$$
$$\frac{d}{dt} \left( \frac{1}{\sqrt{g_{ab}\dot{x}^{a}\dot{x}^{b}}} g_{cb}\dot{x}^{b} \right) - \frac{1}{2\sqrt{g_{ab}\dot{x}^{a}\dot{x}^{b}}} g_{ab,c}\dot{x}^{a}\dot{x}^{b} = 0$$

where

$$g_{ab,c} := \frac{\partial g_{ab}}{\partial x^c} \,. \tag{65}$$

Using the parametrization with arc-length rather than t gives (recall that  $\frac{ds}{dt}=\sqrt{g_{ab}\dot{x}^a\dot{x}^b})$ 

$$\frac{d}{ds}\left(g_{cb}\frac{d\,x^b}{ds}\right) - \frac{1}{2}\,g_{ab,c}\frac{d\,x^a}{ds}\frac{d\,x^b}{ds} = 0\,. \tag{66}$$

Simplifying the above gives,

$$g_{cb}\frac{d^{2}x^{b}}{ds^{2}} + g_{cb,e}\frac{dx^{e}}{ds}\frac{dx^{b}}{ds} - \frac{1}{2}g_{ab,c}\frac{dx^{a}}{ds}\frac{dx^{b}}{ds} = 0$$

$$g_{cb}\frac{d^{2}x^{b}}{ds^{2}} + \frac{1}{2}(g_{cb,e}\frac{dx^{e}}{ds}\frac{dx^{b}}{ds} + g_{ce,b}\frac{dx^{b}}{ds}\frac{dx^{e}}{ds} - g_{ab,c}\frac{dx^{a}}{ds}\frac{dx^{b}}{ds}) = 0$$

$$g_{cb}\frac{d^{2}x^{b}}{ds^{2}} + \frac{1}{2}(g_{cb,e} + g_{ce,b} - g_{eb,c})\frac{dx^{b}}{ds}\frac{dx^{e}}{ds} = 0$$

$$g_{cb}\frac{d^{2}x^{b}}{ds^{2}} + \Gamma_{bec}\frac{dx^{b}}{ds}\frac{dx^{e}}{ds} = 0,$$
(67)

where

$$\Gamma_{ebc} = \frac{1}{2} (g_{cb,e} + g_{ce,b} - g_{eb,c})$$
(68)

We denote with  $g^{bc}$  the matrix inverse of the matrix  $g_{ab}$  so that

$$g_{cb}g^{cf} = \delta_b^f \,. \tag{69}$$

Then multiplying Eq.(67) by the inverse of  $g_{ab}$  we get,

$$g^{cf} g_{cb} \frac{d^2 x^b}{ds^2} + g^{cf} \Gamma_{bcc} \frac{d x^b}{ds} \frac{d x^e}{ds} = 0$$
  
$$\delta^f_b \frac{d^2 x^b}{ds^2} + g^{cf} \Gamma_{bcc} \frac{d x^b}{ds} \frac{d x^e}{ds} = 0$$
  
$$\frac{d^2 x^f}{ds^2} + \Gamma^f_{bc} \frac{d x^b}{ds} \frac{d x^e}{ds} = 0$$
  
. (70)

where the Christoffel symbols are given by:

$$\Gamma_{be}^{f} = g^{cf} \Gamma_{bec} = \frac{g^{cf}}{2} (g_{cb,e} + g_{ce,b} - g_{eb,c})$$
(71)

#### Covariant derivative of a 1-form:

Given a one form  $\omega = \omega_a e^a$  we can let it act on a vector A to obtain the scalar  $\omega_a A^a$  whose covariant derivative is just the ordinary derivative,

$$\nabla_{b}(\omega_{a}A^{a}) = (\omega_{a}A^{a})_{,b},$$

$$A^{a}\nabla_{b}\omega_{a} + \omega_{a}\nabla_{b}A^{a} = \omega_{a,b}A^{a} + \omega_{a}A^{a}_{,b}$$

$$A^{a}\nabla_{b}\omega_{a} = \omega_{a,b}A^{a} + \omega_{a}A^{a}_{,b} - \omega_{a}(A^{a}_{,b} + \Gamma^{a}_{bc}A^{c})$$

$$A^{a}\nabla_{b}\omega_{a} = A^{a}(\omega_{a,b} - \Gamma^{c}_{ba}\omega_{c})$$
(72)

since this should be true for all  ${\cal A}$  therefore we get

$$\nabla_b \omega_a = \omega_{a,b} - \Gamma^c_{ba} \omega_c \tag{73}$$

#### Covariant derivative of a tensor:

Given a tensor  $T_{ab}$  we can construct a scalar by letting it act on two vectors  $T_{ab}A^aB^b$  so that

$$\nabla_c (T_{ab} A^a B^b) = (T_{ab} A^a B^b)_{,c} \tag{74}$$

from which it follows that

$$\nabla_c T_{ab} = T_{ab,c} - \Gamma^d_{ca} T_{db} - \Gamma^d_{cb} T_{ad} \tag{75}$$

similarly for a tensor with upper indices

$$\nabla_c T^{ab} = T^{ab}_{,c} + \Gamma^a_{cd} T^{db} + \Gamma^b_{cd} T^{ad} \tag{76}$$

#### Lecture III: Parallel Transport

We saw in the previous lecture geodesics are curves for which the covariant derivative of the tangent vector in the direction of the tangent vector is zero. Which is to say that the tangent vector is constant along the curve. Consider a curve  $\gamma(s)$  and its tangent vector V. We can consider the equation

$$\nabla_V W = 0 \qquad V^a \Big( W^b_{,a} + W^c \Gamma^b_{ca} \Big) = 0 \qquad (77)$$

Given the vector W at an initial point  $p_0$  we can solve the above equation to obtain the value of W at any point along the curve. The vector W along the curve is then said to be parallel transported from its initial value at  $p_0$ . It is called parallel transport since the equation guarantees that the vector W remains constant and therefore "parallel" to its initial value at  $p_0$ . Parallel transport is a way of comparing vectors at two different points on the manifold by bringing them to the same point so that they lie in the same tangent plane where they can be compared to see if they are parallel. For this reason it is also called connection since it allows a vector to be moved from one tangent space to another keeping it locally parallel to itself.

**PROPERTIES OF THE COVARIANT DERIVATIVE:** The covariant derivative is a linear operator and satisfies:

$$\nabla_V (fW) = \nabla_V (f)W + f\nabla_V W \quad \text{where} \quad (78)$$
$$\nabla_V f = V^a f_{,a} \qquad f_{,a} = \frac{\partial f}{\partial u^a}$$

The inner product between the two vectors remains invariant under parallel transport. To see this consider a curve  $\gamma(s)$  and two vectors A and B defined along the curve such

$$\nabla_V A = \nabla_V B = 0 \tag{79}$$

i.e., they are defined along the curve by parallel transporting them from some initial value at  $p_0$ . The vector V is the tangent vector along the curve. Then the inner product between the two vectors is a function defined along  $\gamma(s)$  and we can consider how this functions changes as we move on the curve  $(V^c = \frac{dU^c}{ds})$ :

$$\frac{d}{ds} \left( g_{ab} A^a B^b \right) = \frac{d g_{ab}}{ds} A^a B^b + g_{ab} \frac{dA^a}{ds} B^b + g_{ab} A^a \frac{dB^b}{ds} \quad (80)$$

$$= g_{ab,c} V^c A^a B^b + g_{ab} \left( -A^d \Gamma^a_{dc} V^c \right) B^b$$

$$+ g_{ab} A^a \left( -B^d \Gamma^b_{dc} V^c \right)$$

$$= V^c A^a B^b \left( g_{ab,c} - g_{db} \Gamma^d_{ac} - g_{ad} \Gamma^d_{bc} \right)$$

using the definition of the Christoffel symbols in terms of the metric

$$\Gamma_{ab}^{d} = \frac{1}{2} g^{dc} \Big( g_{cb,a} + g_{ac,b} - g_{ab,c} \Big)$$

$$g_{dc} \Gamma_{ab}^{d} = \frac{1}{2} \Big( g_{cb,a} + g_{ac,b} - g_{ab,c} \Big)$$
(81)

we get

$$\frac{d}{ds} \left( g_{ab} A^{a} B^{b} \right) = A^{a} B^{b} V^{c} \left( g_{ab,c} - g_{db} \Gamma^{d}_{ac} - g_{ad} \Gamma^{d}_{bc} \right) \qquad (82)$$

$$= A^{a} B^{b} V^{c} \left( g_{ab,c} - \frac{1}{2} \left[ g_{bc,a} + g_{ab,c} - g_{ac,b} \right] - \frac{1}{2} \left[ g_{ac,b} + g_{ba,c} - g_{bc,a} \right] \right) = 0$$

Thus the inner product between vectors do not change as they are parallel transported from one point to another. So far we have only talked about how the covariant derivative acts on vectors. It can, however, be extended and make to act on tensors as well. The above statement about inner product not changing then simply becomes the statement that the metric as a tensor is covariantly constant i.e.,

$$\nabla_a \,\mathbf{g} = 0 \tag{83}$$

where for  $p \in S$  we have  $\mathbf{g}|_p : T_p S \times T_p S \mapsto \mathbb{R}$  such that

$$\mathbf{g}|_{p}\Big(W_{1}, W_{2}\Big) = g_{ab}(p)W_{1}^{a}W_{2}^{b} \qquad W_{1}, W_{2} \in T_{p}S.$$
(84)

**Example:** Consider  $S^2$  with coordinates  $(\theta, \phi)$  and the usual metric:

$$ds^{2} = dx^{2} + dy^{2} + dz^{2}$$

$$= R^{2} d\theta^{2} + R^{2} \sin^{2}(\theta) d\phi^{2}$$

$$g_{ab} = \begin{pmatrix} R^{2} & 0 \\ 0 & R^{2} \sin^{2}\theta \end{pmatrix} \qquad g^{ab} = \begin{pmatrix} R^{-2} & 0 \\ 0 & \frac{1}{R^{2} \sin^{2}\theta} \end{pmatrix}$$
(85)

The non-vanishing Christoffel symbols are:

$$\Gamma_{22}^{1} = -\sin(\theta)\cos(\theta)$$

$$\Gamma_{12}^{2} = \Gamma_{21}^{2} = \cot(\theta)$$
(86)

Let us take a curve  $\gamma(s)$  given by

$$\gamma(s) = \{ (f_1(s), f_2(s)) \in \mathbb{S}^2 \}$$
(87)

The equation we have to solve is:

$$\nabla_{\mathbf{t}} W = 0 \tag{88}$$

subject to initial condition  $W(s = 0) = W_0$ . The above equation in component form is: becomes

$$\frac{dW^b}{ds} + W^c \frac{du^a}{ds} \Gamma^b_{ac} = 0$$
(89)

$$\frac{dW^1}{ds} + W^c \frac{du^a}{ds} \Gamma^1_{ac} = 0$$

$$\frac{dW^1}{ds} + W^2 \frac{du^2}{ds} \Gamma^1_{22} = 0$$
(90)

 $\left(\frac{dW^1}{ds} - \sin(\theta)\cos(\theta)W^2\frac{d\phi}{ds} = 0\right)$ 

$$\frac{dW^2}{ds} + W^c \frac{du^a}{ds} \Gamma_{ac}^2 = 0$$

$$\frac{dW^2}{ds} + W^1 \frac{du^2}{ds} \Gamma_{12}^2 + W^2 \frac{du^1}{ds} \Gamma_{21}^2 = 0$$
(91)

$$\left(\frac{dW^2}{ds} + \cot(\theta) \left[ W^1 \frac{d\phi}{ds} + W^2 \frac{d\theta}{ds} \right] = 0 \right)$$

Let us first solve these equations for a curve going from  $(\theta_0, \phi_0)$  to  $(\theta_0, \phi_0 + \Delta \phi)$  along constant  $\theta_0$ . In this case

$$\frac{d\theta}{ds} = 0 \qquad \qquad \frac{d\phi}{ds} = \frac{1}{R\sin(\theta_0)} \tag{92}$$



Figure 3: Parallel transporting a vector from  $(\theta_0, \phi_0)$  to  $(\theta_0 + \Delta \theta, \phi_0 + \Delta \phi)$  along two different paths. The difference between the transported vectors at  $(\theta_0 + \Delta \theta, \phi_0 + \Delta \phi)$  is a measure of curvature.

The equations become

$$\frac{dW^{1}}{ds} - \cos(\theta_{0})R^{-1}W^{2} = 0$$

$$\frac{dW^{2}}{ds} + \frac{\cot(\theta_{0})}{\sin(\theta_{0})}R^{-1}W^{1} = 0$$
(93)

Uncoupling the equations we get:

$$\frac{d^2 W^a}{ds^2} + R^{-2} \cot^2(\theta_0) W^a = 0 \qquad a = 1, 2 \qquad (94)$$
$$W^1(s) = W_0^2 \sin(\theta_0) \sin(ks) + W_0^1 \cos(ks) \qquad k = \cot(\theta_0)/R$$
$$W^2(s) = W_0^2 \cos(ks) - \frac{W_0^1}{\sin(\theta_0)} \sin(ks)$$

Since 
$$\Delta \phi \ll 1$$
 therefore  $\frac{s}{R} = \sin(\theta_0) \Delta \phi \ll 1$  and we get  
 $W^1(\theta_0, \phi_0 + \Delta \phi) = W_0^1 + W_0^2 \sin(\theta_0) \cos(\theta_0) \Delta \phi$  (95)  
 $W^2(\theta_0, \phi_0 + \Delta \phi) = W_0^2 - W_0^1 \cot(\theta_0) \Delta \phi$ 



Now lets transport this vector along constant  $\phi$  curve from  $(\theta_0, \phi_0 + \Delta \phi)$  to  $(\theta_0 + \Delta \theta, \phi + \Delta \phi)$ . In this case we have:

$$\frac{d\phi}{ds} = 0 \qquad \qquad \frac{d\theta}{ds} = R^{-1} \tag{96}$$

The equations in this case become:

$$\frac{dW^{1}}{ds} = 0 \qquad (97)$$

$$\frac{dW^{2}}{ds} + \cot(\theta) W^{2} R^{-1} = 0 \implies \frac{dW^{2}}{d\theta} + \cot(\theta) W^{2} = 0$$

$$W^{1} = W^{1}(\theta_{0}, \phi_{0} + \Delta\phi) \qquad (98)$$

$$W^{2} = W^{2}(\theta_{0}, \phi_{0} + \Delta\phi) \frac{\sin(\theta_{0})}{\sin(\theta)}$$

Thus we get

$$W^{1}(\theta_{0} + \Delta\theta, \phi_{0} + \Delta\phi) = W^{1}(\theta_{0}, \phi_{0} + \Delta\phi) = W^{1}_{0} + W^{2}_{0} \sin(\theta_{0}) \cos(\theta_{0}) \Delta\phi$$
$$W^{2}(\theta_{0} + \Delta\theta, \phi_{0} + \Delta\phi) = W^{2}(\theta_{0}, \phi_{0} + \Delta\phi) \frac{\sin(\theta_{0})}{\sin(\theta_{0} + \Delta\theta)}$$
$$= W^{2}(\theta_{0}, \phi_{0} + \Delta\phi) \left(1 - \cot(\theta_{0})\Delta\theta\right)$$
$$= \left(W^{2}_{0} - W^{1}_{0} \cot(\theta_{0})\Delta\phi\right) \left(1 - \cot(\theta_{0})\Delta\theta\right)$$
$$= W^{2}_{0}(1 - \cot(\theta_{0})\Delta\theta) - W^{1}_{0} \cot(\theta_{0})\Delta\phi(1 - \cot(\theta_{0})\Delta\theta)$$



Thus the above are the components of the vector which is first parallel transported along the constant  $\theta$  curve and then the constant  $\phi$  curve. Now lets try to parallel transport this vector  $W_0$  in the reverse order by first taking it along the constant  $\phi$  curve and then the constant  $\theta$  curve reaching the same point.

We already know how the components of the vector transform given

by Eq.(98):  

$$\widetilde{W}^{1}(\theta_{0} + \Delta\theta, \phi_{0}) = W_{0}^{1} \qquad (99)$$

$$\widetilde{W}^{2}(\theta_{0} + \Delta\theta, \phi_{0}) = W_{0}^{2}(1 - \cot(\theta_{0})\Delta\theta)$$

$$(\theta_{0} + \Delta\theta, \phi_{0})$$

$$\uparrow \qquad (\theta_{0}, \phi_{0})$$

Now transport this along the constant  $\theta$  curve from  $(\theta_0 + \Delta\theta, \phi_0)$  to  $(\theta_0 + \Delta\theta, \phi_0 + \Delta_0)$ . The equation for this is given by Eq.(95):  $\widetilde{W}^1(\theta_0 + \Delta\theta, \phi_0 + \Delta\phi) = \widetilde{W}^1(\theta_0 + \Delta\theta, \phi_0)$  (100)  $+\widetilde{W}^2(\theta_0 + \Delta\theta, \phi_0) \sin(\theta_0 + \Delta\theta) \cos(\theta_0 + \Delta\theta)\Delta\phi$   $= W_0^1 + W_0^2 \Delta\phi \left(\sin(\theta_0)\cos(\theta_0) - \sin^2(\theta_0)\Delta\theta\right)$   $\widetilde{W}^2(\theta_0 + \Delta\theta, \phi_0 + \Delta\phi) = \widetilde{W}^2((\theta_0 + \Delta\theta, \phi_0))$   $-\widetilde{W}^1((\theta_0 + \Delta\theta, \phi_0)\cot(\theta_0 + \Delta\theta)\Delta\phi)$   $= W_0^2(1 - \cot(\theta_0)\Delta\theta) - W_0^1\cot(\theta_0)\Delta\phi$  $+ \frac{W_0^1}{\sin^2(\theta_0)}\Delta\theta\Delta\phi$ 



$$\widetilde{W}^{1}(\theta_{0} + \Delta\theta, \phi_{0} + \Delta\phi) - W^{1}(\theta_{0} + \Delta\theta, \phi_{0} + \Delta\phi) = -W_{0}^{2}\sin^{2}(\theta_{0})\,\Delta\theta\Delta\phi$$
$$\widetilde{W}^{2}(\theta_{0} + \Delta\theta, \phi_{0} + \Delta\phi) - W^{2}(\theta_{0} + \Delta\theta, \phi_{0} + \Delta\phi) = W_{0}^{1}\,\Delta\theta\Delta\phi$$

A easier way to obtain the above result is to realize that the vectors at near by points differs by covariant derivatives:

$$W(\theta + \Delta\theta, \phi + \Delta\phi) = W(\theta, \phi + \Delta\phi) + \Delta\theta\nabla_{\theta}W(\theta, \phi + \Delta\phi)(101)$$
  
=  $W(\theta, \phi) + \Delta\phi\nabla_{\phi}W(\theta, \phi) + \Delta\theta\nabla_{\theta}W$   
 $+\Delta\theta\Delta\phi\nabla_{\theta}\nabla_{\phi}W(\theta, \phi)$   
 $\widetilde{W}(\theta + \Delta\theta, \phi + \Delta\phi) = \widetilde{W}(\theta + \Delta\theta, \phi) + \Delta\phi\nabla_{\phi}\widetilde{W}(\theta + \Delta\theta, \phi)$   
=  $W(\theta, \phi) + \Delta\phi\nabla_{\phi}W(\theta, \phi) + \Delta\theta\nabla_{\theta}W(\theta, \phi)$   
 $+\Delta\theta\Delta\phi\nabla_{\theta}\nabla_{\phi}W(\theta, \phi)$ 

Define

$$\Delta W = \widetilde{W}(\theta + \Delta \theta, \phi + \Delta \phi) - W(\theta + \Delta \theta, \phi + \Delta \phi)$$

then from Eq(101)

$$\Delta W = \Delta \theta \Delta \phi [\nabla_{\phi}, \nabla_{\theta}] W = \Delta \theta \Delta \phi [\nabla_2, \nabla_1] W \qquad (102)$$

The Riemann curvature tensor is defined as:

$$[\nabla_c, \nabla_d] W^a = R^a_{bcd} W^b \tag{103}$$

 $R^a_{bcd}$  is called the Riemann curvature tensor and it can calculated using the definition of the covariant derivative and is given by:

$$R^{a}_{bcd} = \partial_{c}\Gamma^{a}_{bd} - \partial_{d}\Gamma^{a}_{bc} + \Gamma^{a}_{ce}\Gamma^{e}_{bd} - \Gamma^{a}_{de}\Gamma^{e}_{bc}$$
(104)  
$$R_{abcd} = g_{ae}R^{e}_{bcd}$$

The Riemann curvature tensor is antisymmetric in the first two and the last two indices. For indices taking only two values we get:

$$R_{1111} = R_{1112} = R_{1121} = R_{1122} = R_{2211} = R_{2212} = R_{2221} = R_{2222} = 0$$
$$R_{1211} = R_{2111} = R_{1222} = R_{2122} = 0$$

The four non-vanishing components are:

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} = R^2 \sin^2(\theta_0)$$
(105)

$$R_{1212} = g_{1e}R_{212}^e = g_{11}R_{212}^1 = g_{11}\left(\Gamma_{22,1}^1 - \Gamma_{21,2}^1 + \Gamma_{1f}^1\Gamma_{22}^f - \Gamma_{2f}^1\Gamma_{21}^f\right)$$
  
=  $g_{11}\left(\Gamma_{22,1}^1 - \Gamma_{22}^1\Gamma_{21}^2\right)$   
=  $R^2\left((-\sin(\theta)\cos(\theta))_{,\theta} + \sin(\theta)\cos(\theta)\cot(\theta)\right)$   
=  $R^2\left(-\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta)\right) = R^2\sin^2(\theta)$ 

Thus the four non-vanishing components are:

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} = R^2 \sin^2(\theta_0)$$
(106)

$$\begin{aligned} [\nabla_{2}, \nabla_{1}]W^{1} &= R^{1}_{b21}W^{b} = g^{1a}R_{ab21}W^{b} \qquad (107) \\ &= g^{11}R_{1b21}W^{b} = g^{11}R_{1221}W^{2} = -g^{11}R_{1212}W^{2} \\ &= -R^{-2} \times R^{2}\sin^{2}(\theta_{0})W^{2} = -\sin^{2}(\theta_{0})W^{2} \\ [\nabla_{2}, \nabla_{1}]W^{2} &= R^{2}_{b21}W^{b} = g^{2a}R_{ab21}W^{b} \\ &= g^{22}R_{2b21}W^{b} = g^{22}R_{2121}W^{1} = \frac{1}{R^{2}\sin^{2}(\theta_{0})}R^{2}\sin^{2}(\theta_{0})W^{1} \\ &= W^{1} \end{aligned}$$

Lecture IV: The Riemann Curvature Tensor

In the previous lecture we defined the Riemann curvature tensor using the covariant derivative as:

$$[\nabla_c, \nabla_d] W^a = R^a_{\ bcd} W^b \tag{108}$$

where

$$R^{a}_{bcd} = \partial_{c}\Gamma^{a}_{bd} - \partial_{d}\Gamma^{a}_{bc} + \Gamma^{a}_{ce}\Gamma^{e}_{bd} - \Gamma^{a}_{de}\Gamma^{e}_{bc}$$
(109)

$$R_{abcd} = g_{ae} R^e_{bcd}$$

$$\mathbf{R}_p: T_pM \times T_pM \times T_pM \mapsto T_pM \tag{110}$$

#### Some comments about tangent vectors

Recall that we defined the tangent vectors given by  $\mathbf{r}(u^1, u^2, \cdots, u^n)$  by

$$\mathbf{e}_a = \frac{\partial \mathbf{r}}{\partial u^a}$$
  $a = 1, 2, \cdots, n$ 

This definition is not satisfactory since  $\mathbf{r}$  is an externally defined quantity and not something defined within the manifold. We instead identity the vector with the derivative operator in that particular direction. In  $\mathbb{R}^n$  this is the correspondence:

$$\mathbf{v} \longrightarrow \mathbf{v} \cdot \nabla = v^1 \frac{\partial}{\partial x^1} + \dots + v^n \frac{\partial}{\partial x^n}$$
 (111)

Similarly we define the basis of "tangent vectors" to be given by:

$$\mathbf{e}_a = \frac{\partial}{\partial u^a} \tag{112}$$

Given a curve  $\gamma : [-\epsilon, +\epsilon] \mapsto M$  parametrized by t, the tangent vector at a point  $p = \gamma(0)$  of the curve is given by:

$$\frac{d}{dt}|_{t=0} = \frac{du^a}{dt}|_p \frac{\partial}{\partial u^a}$$
(113)

The numbers  $\frac{du^a}{dt}|_p$  are the components of the tangent vector at p. Under a coordinate transformation  $u^a \mapsto u^{a'}(u^1, \dots, u^n)$  the basis vectors transform in the following way:

$$\mathbf{e}_{a'} = \frac{\partial}{\partial u^{a'}} = \frac{\partial u^a}{\partial u^{a'}} \frac{\partial}{\partial u^a}$$

$$= J_{a'}^{\ a} \mathbf{e}_a , \qquad J_{a'}^{\ a} = \frac{\partial u^a}{\partial u^{a'}}$$
(114)

If W is a tangent vector then the components of W change under a change in the coordinate system:

$$W = W^{a} \mathbf{e}_{a} = W^{a'} \mathbf{e}_{a'} = W^{a'} J^{a}_{a'} \mathbf{e}_{a}$$
(115)  
$$W^{a} = W^{a'} J^{a}_{a'}$$

**Example:** In  $\mathbb{R}^2$  the relation between the Cartesian coordinates  $(x, y) = (u^1, u^2)$  and the polar coordinates  $(r, \theta) = (u^{1'}, u^{2'})$  is given

$$x = r \cos(\theta), \quad y = r \sin(\theta)$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(\frac{y}{x})$$
(116)

$$J_{a'}^{a} = \frac{\partial u^{a}}{\partial u^{a'}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{pmatrix}$$
(117)
$$J_{a}^{a'} = \frac{\partial u^{a'}}{\partial u^{a}} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^{2} + y^{2}}} & -\frac{y}{x^{2} + y^{2}} \\ \frac{y}{\sqrt{x^{2} + y^{2}}} & \frac{x}{x^{2} + y^{2}} \end{pmatrix}$$

$$W = W^{1}\mathbf{e}_{1} + W^{2}\mathbf{e}_{2} = W^{1'}\mathbf{e}_{1'} + W^{2'}\mathbf{e}_{2'}$$
(118)  

$$W^{1'} = W^{1}J_{1}^{1'} + W^{2}J_{2}^{1'}$$
  

$$= W^{1}\cos(\theta) + W^{2}\sin(\theta)$$
  

$$W^{2'} = W^{1}J_{1}^{2'} + W^{2}J_{2}^{2'}$$
  

$$= W^{1}(-\frac{\sin(\theta)}{r}) + W^{2}(\frac{\cos(\theta)}{r})$$

#### BASIS CHANGE

Recall that a linear transformation can be expressed as a matrix (set of numbers) if a basis is chosen. Similarly, if we choose a basis of  $T_pM$  then the linear transformation  $\mathbf{R}_p$  is given by the set of numbers  $R^a_{bcd}(p)$ . However, these numbers change under a change of basis of  $T_pM$ . Let U, V and W be three vectors in  $T_pM$ .  $R_p(U, V, W)$  is an

by:

element of  $T_pM$ :

$$\mathbf{R}_{p}(U^{b}\mathbf{e}_{b}, V^{c}\mathbf{e}_{c}, W^{d}\mathbf{e}_{d}) = U^{b}V^{c}W^{d}\mathbf{R}_{p}(\mathbf{e}_{b}, \mathbf{e}_{c}, \mathbf{e}_{d}) \qquad (119)$$
$$= U^{b}V^{c}W^{d}R^{a}_{bcd}(p)\mathbf{e}_{a}$$

Thus defining the dual  $\mathbf{e}^{a}\mathbf{e}_{b} = \delta^{a}_{b}$  where  $e^{a}$  is the basis of the dual space  $T_{p}^{*}M$  we have:

$$R^{a}_{bcd}(p) = \underbrace{\mathbf{e}^{a}}_{dual \ vector} \underbrace{\mathbf{R}_{p}(e_{b}, e_{c}, e_{d})}_{vector}$$
(120)

If we now change the basis  $\mathbf{e}_a \mapsto \mathbf{e}_{a'} = J_{a'}^{\ b} e_b$  then  $\mathbf{e}^{a'} = J_b^{\ a'} \mathbf{e}^b$  such that

$$\delta_{b'}^{a'} = e^{a'} e_{b'} = J_{b}^{a'} J_{b'}^{\ c} e^{b} e_{c} = J_{b}^{a'} J_{b'}^{\ c} \delta_{c}^{b} = J_{b}^{a'} J_{b'}^{\ b}$$
(121)

$$R_{b'c'd'}^{a'}(p) = e^{a'} R_p(e_{b'}, e_{c'}, e_{d'})$$

$$= J_a^{a'} J_{b'}^{\ b} J_{c'}^{\ c} J_{d'}^{\ d} R_{bcd}^{a}(p)$$
(122)

#### COVARIANT DERIVATIVE OF TENSORS

Recall that:

$$\nabla_a \mathbf{e}_b = \Gamma^f_{ab} \mathbf{e}_f \tag{123}$$

Applying the covariant derivative to  $\mathbf{e}^{a}\mathbf{e}_{b} = \delta^{a}_{b}$  we have:

$$\nabla_{c} \left( \mathbf{e}^{a} \mathbf{e}_{b} \right) = 0$$

$$\left( \nabla_{c} \mathbf{e}^{a} \right) \mathbf{e}_{b} + \mathbf{e}^{a} \left( \nabla_{c} \mathbf{e}_{b} \right) = 0$$

$$\left( \nabla_{c} \mathbf{e}^{a} \right) \mathbf{e}_{b} = -\mathbf{e}^{a} \Gamma_{cb}^{f} \mathbf{e}_{f} = -\Gamma_{cb}^{f} \delta_{f}^{a} = -\Gamma_{cb}^{a}$$

$$\left( \nabla_{c} \mathbf{e}^{a} \right) \mathbf{e}_{b} = -\mathbf{e}^{a} \Gamma_{cb}^{f} \mathbf{e}_{f} = -\Gamma_{cb}^{f} \delta_{f}^{a} = -\Gamma_{cb}^{a}$$

$$\left( \nabla_{c} \mathbf{e}^{a} \right) \mathbf{e}_{b} = -\mathbf{e}^{a} \Gamma_{cb}^{f} \mathbf{e}_{f}$$

Thus we see that:

$$\nabla_c \mathbf{e}^a = -\Gamma^a_{cb} \mathbf{e}^b \tag{125}$$

Thus the covariant derivative of a dual vector  $W = W_b \mathbf{e}^b$  is given by

$$\nabla_a W = \left( W_{b,a} - W_c \Gamma_{ba}^c \right) \mathbf{e}^b \tag{126}$$

The covariant derivative of the Riemann curvature tensor is given by:

$$\nabla_{e} \mathbf{R} = \nabla_{a} \left( R^{a}_{bcd} \mathbf{e}_{a} \mathbf{e}^{b} \mathbf{e}^{c} \mathbf{e}^{d} \right)$$

$$= \left( R^{a}_{bcd,e} + R^{f}_{bcd} \Gamma^{a}_{ef} - R^{a}_{fcd} \Gamma^{f}_{eb} - R^{a}_{bfd} \Gamma^{f}_{ec} - R^{a}_{bcf} \Gamma^{f}_{ed} \right) \mathbf{e}_{a} \mathbf{e}^{b} \mathbf{e}^{c} \mathbf{e}^{d}$$

$$(127)$$

Best way to keep track of indices is the Penrose's abstract index notation (for more details see the book by Asghar Qadir).

Symmetries of the Riemann Curvature Tensor

$$R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}$$
$$R_{abcd} + R_{adbc} + R_{acdb} = 0$$

$$\nabla_e R_{abcd} + \nabla_a R_{becd} + \nabla_b R_{eacd} = 0$$

The number of independent components of  $R_{abcd}$  are:

$$\frac{n^2(n^2-1)}{12}$$

In four dimensions which will be of interest to us it has 20 components.

**RICCI TENSOR AND THE GAUSSIAN CURVATURE** We can define other geometric quantities using the Riemann tensor:

$$\mathbf{Ricci}(U,V) = \mathbf{e}^{c}\mathbf{R}(U,\mathbf{e}_{c},V)$$
(128)

 $\mathbf{Ricci}(U,V) = R_{ab}U^aV^b \qquad R_{ab} = R^c_{acb}$ 

The Ricci scalar is then defined as:

$$R = g^{ab} R_{ab}$$

#### GEODESIC DEVIATION EQUATION

Consider two nearby geodescis and the vector A pointing from one to the other. The *acceleration* of this vector is a measure of the curvature of the manifold. This acceleration is related to the Riemann curvature tensor.



$$\nabla_{\mathbf{t}} \nabla_{\mathbf{t}} \mathbf{A} = \mathbf{R}(\mathbf{t}, \mathbf{t}, \mathbf{A})$$
$$\left(\nabla_{\mathbf{t}} \nabla_{\mathbf{t}} \mathbf{A}\right)^{a} = R^{a}_{bcd} t^{b} t^{c} A^{d}$$

### EINSTEIN TENSOR

The Einstein tensor is defined as

$$G_{ab} = R_{ab} - \frac{1}{2}R\,g_{ab}$$

The most important property of this is that its divergence is zero:

$$\nabla_a G^{ab} = 0 \qquad \qquad G^{ab} = g^{ac} g^{bd} G_{cd}$$

LECTURE V: THE GEOMETRY OF LORENTZ TRANSFORMATIONS

• Speed of light is constant on all inertial reference frames.



$$x^{2} - c^{2} t^{2} = 0 \qquad x^{\prime 2} - c^{2} t^{\prime 2} = 0 \qquad (129)$$

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

$$= A \begin{pmatrix} x \\ ct \end{pmatrix}$$
(130)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
(131)

•  $det(A) = \pm 1$ 

We consider the set of matrices which are connected with the identity so that det(A) = 1.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{T} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
(132)
$$\begin{pmatrix} a & -c \\ b & -d \end{pmatrix} = \begin{pmatrix} d & -b \\ c & -a \end{pmatrix} \implies d = a, \quad b = c$$
$$ad - bc = 1 \implies a^{2} - c^{2} = 1$$
$$(a, c) = (\cosh(\psi), \sinh(\psi)).$$

$$A = \begin{pmatrix} \cosh(\psi) & \sinh(\psi) \\ \sinh(\psi) & \cosh(\psi) \end{pmatrix}.$$
 (133)

The transformations which connects the coordinates assigned by the two observers is given by:

$$x' = \cosh(\psi) x + \sinh(\psi) ct$$

$$ct' = \sinh(\psi) x + \cosh(\psi) ct$$
(134)

We know that the position of the observer  $\mathcal{O}'$  for the observer  $\mathcal{O}$  is given by x = vt. Since its own position for the observer  $\mathcal{O}'$  is given

by x' = 0. Therefore:

$$0 = \cosh(\psi) vt + \sinh(\psi) ct \implies \tanh(\psi) = -\frac{v}{c} \quad (135)$$
  

$$\cosh(\psi) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \sinh(\psi) = -\frac{\frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$
  

$$x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(x - vt\right), \quad ct' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(ct - \frac{v}{c}x\right)$$

In the limit  $c \mapsto \infty$  we recover the Galilean transformations and Galilean addition of velocities.

ALTERNATIVE DERIVATION



$$\mathcal{O}_1': \qquad x = vt \tag{136}$$

$$\mathcal{O}_2': \qquad x = L + vt \tag{137}$$

$$\mathcal{O}_3': \qquad x = 2L + vt. \tag{138}$$

$$x_1 = L + vt_1, \quad x_1 = ct_1 \tag{139}$$

$$\Rightarrow t_1 = \frac{L}{c-v}, \quad x_1 = \frac{cL}{c-v} \tag{140}$$

$$(ct_1, x_1) = \left(\frac{cL}{c-v}, \frac{cL}{c-v}\right).$$
(141)

**<u>Event B</u>**:  $(ct_2, x_2)$  such that

⇒

$$x_2 = -ct_2 + \alpha, \quad x_2 = 2L + vt \tag{142}$$

where C lies on this line.

$$\Rightarrow \frac{cL}{c-v} = -\frac{cL}{c-v} + \alpha \tag{143}$$

$$\Rightarrow \alpha = \frac{2cL}{c-v} \tag{144}$$

$$\Rightarrow x_2 = -ct_2 + \frac{2cL}{c-v} \tag{145}$$

$$x_2 = 2L + vt_2 \tag{146}$$

$$t_2 = \frac{\left(\frac{2cL}{c-v} - 2L\right)}{c+v} \tag{147}$$

$$= \frac{2Lv}{c^2 - v^2} \tag{148}$$

$$x_2 = -\frac{2vcL}{c^2 - v^2} + \frac{2cL}{c - v} = \frac{-2vcL + 2c^2L + 2vcL}{c^2 - v^2}$$
(149)

$$= \frac{2c^2L}{c^2 - v^2}, \quad ct_2 = \frac{v}{c}x_2. \tag{150}$$

Event A and B define the x'-axis of  $\mathcal{O}'$  and its equation is  $ct = \frac{v}{c}x$ . Thus when x = vt,  $\Rightarrow x' = 0$ , and when  $ct = \frac{v}{c}x \Rightarrow ct' = 0$ .

$$\Rightarrow x' = \gamma_1(x - vt) \tag{151}$$

$$ct' = \gamma_2 \left( ct - \frac{v}{c} x \right), \tag{152}$$

where  $\gamma_1$  and  $\gamma_2$  are functions of v. Since

$$x = ct \quad \Rightarrow x' = ct' \tag{153}$$

$$\left\{ \begin{array}{c} ct' = \gamma_1(ct - vt) \\ ct' = \gamma_2(ct - vt) \end{array} \right\} \quad \gamma_1 = \gamma_2$$
 (154)

$$x' = \gamma(x - vt) \tag{155}$$

$$ct' = \gamma \left( ct - \frac{v}{c} x \right) \qquad \gamma_1 = \gamma_2 = \gamma.$$
 (156)

Introducing a third observer  $\mathcal{O}''$  moving with speed u with respect to  $\mathcal{O}'$ .

$$x'' = \gamma(u)(x' - ut') \tag{157}$$

$$ct'' = \gamma(u) \left( ct' - \frac{u}{c} x' \right) \tag{158}$$

$$x'' = \gamma(u) \left[ \gamma(v)(x - vt) - \frac{u}{c} \gamma(v) \left( ct - \frac{v}{c} x \right) \right]$$
(159)

$$= \gamma(u)\gamma(v)\left[x - vt - \frac{u}{c}ct + \frac{uv}{c^2}x\right]$$
(160)

$$= \gamma(u)\gamma(v)\left[\left(1+\frac{uv}{c^2}\right)x - (u+v)t\right]$$
(161)

$$= \gamma(u)\gamma(v)\left(1+\frac{uv}{c^2}\right)[x-wt]; \quad w = \frac{u+v}{1+\frac{uv}{c^2}}$$
(162)

$$\gamma(w) = \gamma(u)\gamma(v)\left(1+\frac{uv}{c^2}\right).$$
(163)

If  $u = -v \Rightarrow w = 0$ ,  $\gamma(0) = 1$ .

$$\Rightarrow \gamma(v)\gamma(v)\left(1-\frac{v^2}{c^2}\right) = 1, \quad \gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{164}$$

Thus 
$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - v^2/c^2}}$$
 (165)

and addition of velocities,

$$u, v \rightarrow \frac{u+v}{1+\frac{uv}{c^2}}$$
 (166)

$$\begin{cases} x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{2}}} \\ c t' = \frac{c t - \frac{v}{c} x}{\sqrt{1 - \frac{v^2}{c^2}}} \end{cases}$$
Lorentz transformations (167)

## Length Contraction:

An object with length  $\ell_0$  in O reference frame. This object is observed by O' and he measures both end points at the same time at t' = 0.

$$t_1 - \frac{v}{c^2} x_1 = 0, \qquad t_2 - \frac{v}{c^2} x_2 = 0$$
 (168)

since 
$$x_1 = 0, \quad x_2 = \ell_0$$
 (169)

$$\Rightarrow t_1 = 0, \qquad t_2 = \frac{v}{c^2} \ell_0 \tag{170}$$

$$x_{2}' - x_{1}' = \gamma(x_{2} - x_{1}) - \gamma v(t_{2} - t_{1})$$
(171)

$$= \gamma \ell_0 - \ell_0 \gamma \frac{v^2}{c^2} \tag{172}$$

$$= \gamma \ell_0 \left( 1 - \frac{v^2}{c^2} \right) \tag{173}$$

$$\ell = \ell_0 \sqrt{1 - \frac{v^2}{c^2}}.$$
 (174)

### Time dilation:

Suppose observer O has a clock at x = 0. Two events happen at time  $t_1$  and time  $t_2$ . The difference for both is given by  $t_2 - t_1$ . The time difference as seen by O' is

$$t_2' - t_1' = \gamma(t_2 - t_1) - \gamma \frac{v}{c^2}(x_2 - x_1)$$
(175)

$$= \gamma(t_2 - t_1) \tag{176}$$

$$= \frac{(t_2 - t_1)}{\sqrt{1 - v^2/c^2}} > t_2 - t_1 \tag{177}$$

Thus moving clock appears slow.

# Spacetime distance:

$$-1 < \frac{v}{c} < 1, \qquad \Rightarrow \quad \exists \quad \psi \text{ such that } \tanh \psi = v/c.$$
$$\gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}} = \cosh \psi$$
$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh \psi & -\sinh \psi \\ -\sinh \psi & \cosh \psi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$
since  $\cosh^2 \eta - \sinh^2 \eta = 1, \quad \Rightarrow (ct')^2 - (x')^2 = (ct)^2 - x^2$ 

Thus if we define the spacetime distance between two events with coordinates  $(ct_1, x_1)$  and  $(ct_2, x_2)$  by

$$c^{2}(t_{1}-t_{2})^{2}-(x_{1}-x_{2})^{2}$$
(178)

then the two observers  $\mathcal{O}$  and  $\mathcal{O}'$  will agree on this distance

$$c^{2}(t_{1}-t_{2})^{2}-(x_{1}-x_{2})^{2}=c^{2}(t_{1}'-t_{2}')^{2}-(x_{1}'-x_{2}')^{2}.$$
 (179)



# LORENTZ GROUP IN FOUR DIMENSIONS

The set of  $4 \times 4$  matrices which preserve the quadratic form

$$c^2 t^2 - x^2 - y^2 - z^2$$

form a group known as the Lorentz group O(1,3). Since

$$c^{2}t^{2} - x^{2} - y^{2} - z^{2} = \begin{pmatrix} ct & x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

therefore  $g \in O(1,3)$  is such that

$$g^T \eta \, g = \eta$$

**Generators:** The generators of SO(1,3) are

ROTATIONS

Consider a vector in  $\mathbb{R}^3$ :  $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Rotation preseves the length of the vector

$$\mathbf{r} \mapsto R \, \mathbf{r} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \qquad x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$$

$$R^T R = I \qquad \det(R) = 1$$

$$R = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \in SO(3)$$

Consider  $2 \times 2$  hermitian matrices:

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \qquad a, b, c, d \in \mathbb{C}$$

$$H^{\dagger} := \overline{(H^T)} = H \implies H = \begin{pmatrix} \alpha & \beta + i\gamma \\ \beta - i\gamma & \delta \end{pmatrix} \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

$$H = \frac{\alpha + \delta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_1} + \gamma \underbrace{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}_{\sigma_2} + \frac{\alpha - \delta}{2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_3}$$

 $\mathcal{H} = \{ H \text{ is hermitian} \} \cong \mathbb{R}^4$ 

$$\mathcal{H}_0 = \{ H \text{ is hermitian and } Tr(H) = 0 \} \cong \mathbb{R}^3$$

Vectors in  $\mathbb{R}^3 \iff$  Traceless Hermitian Matrices

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \begin{pmatrix} z & x + iy \\ x - iy & -z \end{pmatrix} = H$$
$$\mathbf{r}^2 = -\det(H)$$
$$\mathbf{r} \mapsto R \mathbf{r} \iff H \mapsto U H U^{-1}$$
$$R^T R = I \quad \det(R) = 1 \quad \Longleftrightarrow \quad U^{\dagger} U = I \quad \det(U) = 1$$
$$R \in SO(3) \iff U \in SU(2)$$
$$SO(3) \cong SU(2)/\mathbb{Z}_2$$

$$R(\mathbf{n}, \theta) \qquad \mapsto \qquad U(\mathbf{n}, \theta) = \exp\left(i\frac{\theta}{2}\,\mathbf{n}\cdot\vec{\sigma}\right)$$

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \mapsto \begin{pmatrix} Re(a^2 - b^2) & Im(a^2 + b^2) & -2Re(a b) \\ -Im(a^2 - b^2) & Re(a^2 + b^2) & 2Im(a b) \\ 2Re(ab^*) & 2Im(ab^*) & |a|^2 - |b|^2 \end{pmatrix}$$

$$\mathcal{H} \cong \mathbb{R}^{4}$$
$$\mathbf{r} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad \mapsto \qquad \begin{pmatrix} ct + z & x + iy \\ x - iy & ct - z \end{pmatrix} = H = ct I + \mathbf{r} \cdot \vec{\sigma}$$

$$-\det(H) = -(c t)^2 + x^2 + y^2 + z^2$$

$$\mathbf{r} \mapsto L \, \mathbf{r} \quad \text{such that} \quad -(c \, t)^2 + x^2 + y^2 + z^2 \text{ is unchanged}$$
$$L^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

 $L \in O(3,1)$ 

Elements of O(3, 1) which can be continuously connected with identity determinant 1 form a subgroup of  $O(3, 1)=SO_+(3, 1)$ 

• 
$$\dim O(3,1) = \dim SO_+(3,1) = 6$$

In  $\mathbb{R}^4$  there are six planes which can be rotated independently.

 $\mathbf{r} \mapsto L \, \mathbf{r} \qquad L \in SO_+(3,1)$ 

$$\begin{pmatrix} ct+z & x+iy\\ x-iy & ct-z \end{pmatrix} = H \qquad \mapsto \qquad A H A^{\dagger}$$

• A is an arbitrary complex matrix with det(A) = 1

$$-\det(H) \qquad \mapsto \qquad -\det(H)$$
$$L \in SO_{+}(3,1) \qquad \qquad A \in SL(2,\mathbb{C})$$

$$A = \exp\left(i\frac{\theta}{2}\mathbf{n}\cdot\vec{\sigma} - \frac{\psi}{2}\mathbf{u}\cdot\vec{\sigma}\right)$$

Rotation:  $\theta$ , **n** Boost:  $\psi$ , **u** 

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \\ \begin{pmatrix} \frac{1}{2}(|a|^2 + |b|^2 + |c|^2 + |d|^2) & -Re(ab^* + cd^*) & Im(ab^* + cd^*) & \frac{1}{2}(|a|^2 - |b|^2 + c|^2 - |d|^2) \\ -Re(a^*c + b^*d) & Re(a^*d + b^*c) & -Im(ad^* - bc^*) & -Re(a^*c - b^*d) \\ Im(a^*c + b^*d) & -Im(a^*d + b^*c) & Re(ad^* - bc^*) & Im(a^*c - b^*d) \\ \frac{1}{2}(|a|^2 + |b|^2 - |c|^2 - |d|^2) & -Re(ab^* - cd^*) & Im(ab^* - cd^*) & \frac{1}{2}(|a|^2 - |b|^2 - |c|^2 + |d|^2) \end{pmatrix}$$

The natural action of  $2 \times 2$  matrices is on a two dimensional vector space:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad \mapsto \qquad A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

These two dimensional complex vectors on which A acts linearly are called spinors

• Spinors are two dimensional representation of  $SL(2, \mathbb{C})$ . If you think of  $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$  as projective coordinates on a sphere, then on  $z = \frac{z_1}{z_2}$  the  $SL(2, \mathbb{C})$  acts as



These generators satisfy the following commutation relations

$$[J_a, J_b] = \epsilon_{abc} J_c$$

$$[K_a, K_b] = -\epsilon_{abc} J_c$$

$$[K_a, J_b] = \epsilon_{abc} K_c$$
(181)

If we define new generators  $A_a = \frac{K_a + iJ_a}{2}$  and  $B_a = \frac{-K_a + iJ_a}{2}$  then

$$[A_a, A_b] = i\epsilon_{abc}A_c, \ [B_a, B_b] = i\epsilon_{abc}B_c, \ [A_a, B_b] = 0$$
(182)

Thus the new generators  $A_a$  and  $B_b$  each satisfy the angular momentum commutation relation and commute with each other. Thus we can use the result of the angular momentum commutation relation derived earlier and label the states with two angular momentum quantum numbers, one corresponding to  $A^2$ ,  $A_3$  and other corresponding to  $B^2$ ,  $B_3$ ,  $j_1$  and  $j_2$ . Thus representations of the Lorentz group are labeled by two quantum numbers  $(j_1, j_2)$  with  $j_{1,2} \in \{0, , \frac{1}{2}, 1, \frac{3}{2}, \dots\}$ .

- $(j_1, j_2) = (0, 0)$  is the Lorentz scalar
- $(j_1, j_2) = (\frac{1}{2}, 0)$  is the chiral 2-component spinor
- $(j_1, j_2) = (\bar{0}, \frac{1}{2})$  is also chiral 2-component spinor
- $(j_1, j_2) = (\frac{1}{2}, \frac{1}{2})$  is the 4-vector
- $(j_1, j_2) = (1, 0)$  is the self-dual 2-form,  $F^+_{\mu\nu}$
- $(j_1, j_2) = (0, 1)$  is the antiself-dual 2-form  $F^-_{\mu\nu}$
- $(j_1, j_2) = (1, 1)$  is the traceless part of the metric  $g_{\mu\nu}$

# $\mathcal{H}_0$ Spin 0:

It is one dimensional with basis vector  $|0,0\rangle$ . Thus the operators are

all numbers  $(1 \times 1 \text{ matrices})$ :

$$\widehat{J}^2 \xrightarrow{\{|0,0\rangle} (0), \ \widehat{J}_a \xrightarrow{\{|0,0\rangle} (0), \ a = 1, 2, 3.$$
 (183)

# $\mathcal{H}_{\frac{1}{2}}$ Spin $\frac{1}{2}$

It is two dimensional with basis vectors  $\{|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle\}$ . The operators are now 2 × 2 matrices:

$$\widehat{J}^{2} \xrightarrow{\{|\frac{1}{2},\frac{1}{2}\rangle,|\frac{1}{2},-\frac{1}{2}\rangle\}} \begin{pmatrix} \frac{1}{2}(\frac{1}{2}+1) & 0\\ 0 & \frac{1}{2}(\frac{1}{2}+1) \end{pmatrix}$$

$$\widehat{J}_{3} \xrightarrow{\{|\frac{1}{2},\frac{1}{2}\rangle,|\frac{1}{2},-\frac{1}{2}\rangle\}} \begin{pmatrix} \frac{1}{2} & 0\\ 0 & -\frac{1}{2} \end{pmatrix} = \frac{\sigma_{3}}{2}$$

$$\widehat{J}_{1} \xrightarrow{\{|\frac{1}{2},\frac{1}{2}\rangle,|\frac{1}{2},-\frac{1}{2}\rangle\}} \begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix} = \frac{\sigma_{1}}{2}, \quad \widehat{J}_{2} \xrightarrow{\{|\frac{1}{2},\frac{1}{2}\rangle,|\frac{1}{2},-\frac{1}{2}\rangle\}} \begin{pmatrix} 0 & i\\ -i & 0 \end{pmatrix} = \frac{\sigma_{2}}{2}$$

In the problem set 1 we saw that the Pauli matrices satisfy the following commutation relations

$$\left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2}\right] = i\epsilon_{abc}\frac{\sigma_c}{2}$$

### $\mathcal{H}_1$ Spin 1

It is three dimensional with basis vectors  $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ . The

operators are now  $3 \times 3$  matrices:

$$\widehat{J}^{2} \xrightarrow{\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}} \begin{pmatrix} 1(1+1) & 0 & 0 \\ 0 & 1(1+1) & 0 \\ 0 & 0 & 1(1+1) \end{pmatrix}$$
(185)
$$\widehat{J}_{3} \xrightarrow{\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\widehat{J}_{1} \xrightarrow{\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \widehat{J}_{2} \xrightarrow{\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

# $\mathcal{H}_{\frac{3}{2}}$ Spin $\frac{3}{2}$

It is four dimensional with basis vectors  $\{|\frac{3}{2},\frac{3}{2}\rangle, |\frac{3}{2},\frac{1}{2}\rangle, |\frac{3}{2},-\frac{1}{2}\rangle, |\frac{3}{2},\frac{3}{2}\}.$ 

The operators are now  $4 \times 4$  matrices

$$\begin{split} \widehat{J}^2 & \{|\frac{3}{2},\frac{3}{2}\rangle, |\frac{3}{2},\frac{1}{2}\rangle, |\frac{3}{2},-\frac{1}{2}\rangle, |\frac{3}{2},\frac{3}{2}\rangle\} \begin{pmatrix} \frac{3}{2}(\frac{3}{2}+1) & 0 & 0 \\ 0 & \frac{3}{2}(\frac{3}{2}+1) & 0 \\ 0 & 0 & \frac{3}{2}(\frac{3}{2}+1) & 0 \\ 0 & 0 & 0 & \frac{3}{2}(\frac{3}{2}+1) \end{pmatrix} \\ \widehat{J}_3 & \{|\frac{3}{2},\frac{3}{2}\rangle, |\frac{3}{2},\frac{1}{2}\rangle, |\frac{3}{2},-\frac{1}{2}\rangle, |\frac{3}{2},\frac{3}{2}\rangle\} \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \\ \widehat{J}_1 & \{|\frac{3}{2},\frac{3}{2}\rangle, |\frac{3}{2},\frac{1}{2}\rangle, |\frac{3}{2},-\frac{1}{2}\rangle, |\frac{3}{2},\frac{3}{2}\rangle\} \\ \widehat{J}_2 & \{|\frac{3}{2},\frac{3}{2}\rangle, |\frac{3}{2},\frac{1}{2}\rangle, |\frac{3}{2},-\frac{1}{2}\rangle, |\frac{3}{2},\frac{3}{2}\rangle\} \\ \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \\ \widehat{J}_2 & \{|\frac{3}{2},\frac{3}{2}\rangle, |\frac{3}{2},\frac{1}{2}\rangle, |\frac{3}{2},-\frac{1}{2}\rangle, |\frac{3}{2},\frac{3}{2}\rangle\} \\ \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}. \end{split}$$

Notice that in each of the above case the matrices satisfy the same commutation relation and that not more than one matrix is diagonal (since otherwise commutation relation will not be satisfied).

# Exercise: Construct Spin 2 matrices.

## Spinor representation: Chiral, Dirac and Majorana

Chiral 2-component spinor  $(\frac{1}{2}, 0)$  transform in an irreducible repre-

sentation of the Lorentz group. Acting on this 2-component spinor

$$A^{a} = \frac{\sigma^{a}}{2}, \quad B^{a} = 0 \Rightarrow J_{a} = -i\frac{\sigma_{a}}{2}, \quad K_{a} = \frac{\sigma_{a}}{2}$$
$$\psi_{L} = \begin{pmatrix} \psi_{1} \\ \psi_{2} \end{pmatrix} \mapsto_{\text{rotation}} e^{-i\theta \hat{n} \cdot \frac{\sigma}{2}} \begin{pmatrix} \psi_{1} \\ \psi_{2} \end{pmatrix} \qquad (186)$$
$$\psi_{L} = \begin{pmatrix} \psi_{1} \\ \psi_{2} \end{pmatrix} \mapsto_{\text{boost}} e^{\beta \hat{n} \cdot \frac{\sigma}{2}} \begin{pmatrix} \psi_{1} \\ \psi_{2} \end{pmatrix}$$

Chiral 2-component spinor  $(0, \frac{1}{2})$  also transform in an irreducible representation of the Lorentz group. Acting on this 2-component spinor

$$A^{a} = 0, \quad B^{a} = \frac{\sigma^{a}}{2} \Rightarrow J_{a} = -i\frac{\sigma_{a}}{2}, \quad K_{a} = -\frac{\sigma_{a}}{2}$$
$$\psi_{R} = \begin{pmatrix} \psi_{1} \\ \psi_{2} \end{pmatrix} \mapsto_{\text{rotation}} e^{-i\theta \,\hat{n} \cdot \frac{\sigma}{2}} \begin{pmatrix} \psi_{1} \\ \psi_{2} \end{pmatrix} \qquad (187)$$
$$\psi_{R} = \begin{pmatrix} \psi_{1} \\ \psi_{2} \end{pmatrix} \mapsto_{\text{boost}} e^{-\beta \,\hat{n} \cdot \frac{\sigma}{2}} \begin{pmatrix} \psi_{1} \\ \psi_{2} \end{pmatrix}$$

The Dirac spinor  $\psi_D$  transforms in  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$  representation of the Lorentz group which is a reducible representation:

$$\psi_{D} = \begin{pmatrix} \psi_{L} \\ \psi_{R} \end{pmatrix} \mapsto_{\text{rotation}} \begin{pmatrix} e^{-i\theta \,\hat{n} \cdot \vec{\sigma}} & 0 \\ 0 & e^{-i\theta \,\hat{n} \cdot \vec{\sigma}} \end{pmatrix} \begin{pmatrix} \psi_{L} \\ \psi_{R} \end{pmatrix} (188)$$
$$\psi_{D} = \begin{pmatrix} \psi_{L} \\ \psi_{R} \end{pmatrix} \mapsto_{\text{boost}} \begin{pmatrix} e^{\beta \,\hat{n} \cdot \vec{\sigma}} & 0 \\ 0 & e^{-\beta \,\hat{n} \cdot \vec{\sigma}} \end{pmatrix} \begin{pmatrix} \psi_{L} \\ \psi_{R} \end{pmatrix}$$

Let us defines  $\sigma^{\mu} = (1, \vec{\sigma})$  and  $\bar{\sigma}^{\mu} = (1, -\vec{\sigma})$ . We will use the so called chiral representation of the gamma matrices

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$$
(189)

Then

$$S^{\mu\,\nu} = \frac{1}{4} [\gamma^{\mu}, \gamma^{\nu}]$$

satisfies the same commutation relation as the generators of the Lorentz group:

$$[S^{\mu\nu}, S^{\rho\sigma}] = S^{\mu\sigma}\eta^{\mu\rho} + S^{\rho\mu}\eta^{\nu\sigma} - S^{\nu\sigma}\eta^{\rho\mu} - S^{\rho\nu}\eta^{\sigma\mu}$$

The Lorentz transformation of the 4-component Dirac spinor with parameters  $\omega_{\mu\nu}$  is then given by<sup>1</sup>

$$S = e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}} \tag{190}$$

Where  $\omega_{\mu\nu}$  is the "angle" by which  $x^{\mu} - x^{\nu}$  plane is rotated. Remember that for  $x^0 - x^i$  "rotation" is a boost in the  $x^i$ -direction (a hyperbolic rotation).

$$S_{rot}(\vec{n}) = e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}} = \begin{pmatrix} e^{i\,\vec{n}\cdot\vec{\sigma}/2} & 0\\ 0 & e^{i\,\vec{n}\cdot\vec{\sigma}/2} \end{pmatrix}$$
(191)  
$$S_{boost}(\vec{n}) = e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}} = \begin{pmatrix} e^{\vec{n}\cdot\vec{\sigma}/2} & 0\\ 0 & e^{-\vec{n}\cdot\vec{\sigma}/2} \end{pmatrix}$$

<sup>&</sup>lt;sup>1</sup>Dirac spinor is a 4-component object which transforms as  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ .  $\omega_{\mu\nu}$  is the "angle" of rotation in the  $x^{\mu} - x^{\nu}$  plane.